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A Microscopic Model for a One Parameter Class of Fractional Laplacians with Dirichlet Boundary Conditions

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Abstract

We prove the hydrodynamic limit for the symmetric exclusion process with long jumps given by a mean zero probability transition rate with infinite variance and in contact with infinitely many reservoirs with density α at the left of the system and β at the right of the system. The strength of the reservoirs is ruled by $\kappa N^{-\theta} > 0$. Here N is the size of the system, $\kappa > 0$ and $\theta \in \mathbb{R}$. Our results are valid for $\theta \leq 0$. For $\theta = 0$, we obtain a collection of fractional reaction–diffusion equations indexed by the parameter κ and with Dirichlet boundary conditions. Their solutions also depend on κ. For $\theta < 0$, the hydrodynamic equation corresponds to a reaction equation with Dirichlet boundary conditions. The case $\theta > 0$ is still open. For that reason we also analyze the convergence of the unique weak solution of the equation in the case $\theta = 0$ when we send the parameter κ to zero. Indeed, we conjecture that the limiting profile when $\kappa \to 0$ is the one that we should obtain when taking small values of $\theta > 0$.

1. Introduction

Normal (diffusive) transport phenomena are described by standard random walk models. Anomalous transport, in particular transport phenomena giving rise to superdiffusion, are nowadays encapsulated in the Lévy flights or Lévy walks framework [7,8] and appear in physics, finance and biology. The term "Lévy flight" was coined by Mandelbrot and is nothing but a random walk in which the step-lengths have a probability distribution that is heavy tailed. A (one-dimensional) Lévy walker moves with a constant velocity v for a heavy-tailed random time τ on a distance $x = v\tau$ in either direction with equal probability and then chooses a new direction and moves again. One then easily shows that for Lévy flights or Lévy walks, the space-time scaling limit $P(x, t)$ of the probability distribution of the particle position $x(t)$ is solution of the fractional diffusion equation

$$\partial_t P = -c(-\Delta)^{\gamma/2} P, \tag{1.1}$$

where c is a constant and $\gamma \in (1, 2)$. In physics, the description of anomalous transport phenomena by Lévy walks instead of Lévy flights is sometimes preferred despite the two models having the same scaling limit form provided by (1.1) because the first ones have a finite speed of propagation (see [7] for more details).

While Lévy walks and Lévy flights are today well known and popular models to describe superdiffusion in infinite systems in various application fields, there have recently been several physical studies pointing out that it would be desirable to have a better understanding of Lévy walks in bounded domains. For bounded domains, boundary conditions and exchange with reservoirs or environment have to be taken into account. A particular interest for this problem is related to the description of anomalous diffusion of energy in low-dimensional lattices [9,19] in contact with reservoirs [10,11,18,20]. It is for example argued in [20] that the density profiles of Lévy walkers in a finite box with absorbtion-reflection-creation well reproduces the temperature profile of some chains of harmonic oscillators with conservative momentum-energy noise and thermostat boundaries. It is well established that superdiffusive systems are much more sensitive to the reservoirs and boundaries than diffusive systems but quantitative informations, like the form of the singularities of the profiles at the boundaries, are still missing.

In this work, motivated by these studies, we propose a simple interacting particle system which may be considered as a substitute to Lévy flights in bounded domains with reservoirs when Lévy flights are moreover *interacting*. Indeed, the previous studies consider only non-interacting cases. The system considered here is composed of interacting Lévy flights on a one-dimensional lattice. More exactly, the system is an exclusion process on a finite lattice of size N with jumps having a distribution in the form $p(z) \sim |z|^{-(1+\gamma)}$, with $\gamma > 0$, and in contact with some reservoirs at density α (resp. β) at its left (resp. right boundary). The reservoirs' coupling is modulated by a prefactor $\kappa N^{-\theta}$, $\kappa > 0$, $\theta \in \mathbb{R}$. In this work we focus on the case $\gamma \in (1, 2)$ (the case $\gamma > 2$ was solved in [4]) and we also restrict to the case $\theta \leq 0$. The cases $\theta > 0$, $\gamma \in (0, 1]$ and $\gamma = 2$ remain open.

Our main result is the derivation of the hydrodynamic limit for the density of particles for this system. The limiting PDE depends on the value of κ and takes the form of a fractional heat equation with a singular reaction term, see (2.10).[1] The singular reaction term fixes the density on the left to be α and on the right to be β. In our opinion this singular reaction term, which is due to the presence of the reservoirs, should be considered more as a boundary condition than as a reaction term. We obtain in this way a new family of regional fractional Laplacians on [0, 1] with zero Dirichlet boundary conditions indexed by κ and taking the form

$$\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1, \quad V_1(u) = r^-(u) + r^+(u), \tag{1.2}$$

[1] In the diffusive case $\gamma > 2$ the limiting PDE is given by the heat equation with Dirichlet boundary conditions [4]. It does not depend on κ.

where

$$r^{-}(u) = c_\gamma \gamma^{-1} u^{-\gamma} \quad \text{and} \quad r^{+}(u) = c_\gamma \gamma^{-1} (1-u)^{-\gamma}$$

and c_γ is a constant depending on γ. These operators are symmetric non-positive when restricted to the set of smooth functions compactly supported in (0, 1). For $\kappa = 1$, we recover the so-called restricted fractional Laplacian while in the limit $\kappa \to 0$ we get the so-called regional fractional Laplacian. We recall that since the fractional Laplacian is a non-local operator, the definition of a fractional Laplacian with Dirichlet boundary conditions is not obvious from a modeling point of view. In the PDE's literature several candidates have been proposed, for instance, "restricted fractional Laplacian", "spectral fractional Laplacian", "Neumann Fractional Laplacian" [2,23], but often without a clear physical interpretation. A probabilistic interpretation of these operators is sometimes possible and may enlighten their meaning. The restricted fractional Laplacian ($\kappa = 1$) corresponds to the generator of a γ-Lévy stable process killed outside of (0, 1), while the regional fractional Laplacian ($\kappa = 0$) corresponds to the generator of a censored γ-Lévy stable process on (0, 1) [5,15]. For $\kappa \neq 0, 1$ we could rely on the Feynman–Kac formula but we do not pursue this issue here. As mentioned above our reservoirs are regulated by the parameters $\kappa N^{-\theta}$, $\kappa > 0$ and in this work we focus on the case $\theta \leq 0$. The case $\theta > 0$ is quite interesting and we conjecture that for small values of $\theta > 0$ it is given by (2.10) for the choice $\kappa = 0$. To support this conjecture, in Theorem 2.13, we analyse the convergence of the profile that we obtained for $\theta = 0$ and which is indexed in κ, when $\kappa \to 0$ (we also analyse the case $\kappa \to \infty$ confirming the behaviour obtained from the microscopic system when $\theta < 0$) and indeed, we obtain that the limiting profiles are weak solution of the conjectured equation. We remark that the main problem in analysing the behavior of the microscopic system in this case is at the level of the derivation of the Dirichlet boundary conditions, since the two-blocks estimate does not work. We leave this open problem for a future work. After having obtained the hydrodynamic limits, we have studied their stationary solutions $\bar\rho^\kappa$, which are not explicit apart from the case $\kappa = 1$ and the case $\kappa = \infty$, i.e. $\bar\rho^\infty = \lim_{\kappa\to\infty} \bar\rho^\kappa$. These profiles coincide with the profiles of the microscopic system in their non-equilibrium stationary states (see [3] for the $\kappa = 1$ case). The bounded continuous function $\bar\rho^\kappa$ has α and β as boundary conditions and solves in a distributional sense the equation

$$\mathbb{L}_\kappa \bar\rho^\kappa = -\kappa V_0, \quad V_0(u) = \alpha r^{-}(u) + \beta r^{+}(u). \tag{1.3}$$

There are many recent studies focusing on the regularization properties of fractional operators in bounded domains. Even in this one dimensional setup, the question is in general non trivial. For $\kappa = 1$, $\bar\rho^\kappa$ can be computed explicitly and it appears that it is smooth in the interior of [0, 1] but has only Hölder regularity equal to $\gamma/2$ at the boundaries. For $\kappa \neq 1$, it should be possible to prove the interior regularity of $\bar\rho^\kappa$ by some existing methods [21] but the boundary regularity that numerical simulations seem to indicate depends on κ is much more challenging and seems to be open. We prove that as $\kappa \to 0$, $\bar{\rho}^\kappa \to \bar{\rho}^0$ in a suitable topology and that $\bar{\rho}^0$ is a weakly harmonic function of the regional fractional Laplacian $\mathbb{L}_0$, i.e. we can take $\kappa = 0$ in (1.3). We left these interesting questions for future works.

The paper is organized as follows: in Section 2 we introduce the model and we present all the PDE's that will be related to its hydrodynamic limit. We also present the main results of this work, namely the hydrodynamic limit stated in Theorem 2.12, the convergence, when $\kappa \to 0$ and when $\kappa \to \infty$, of the hydrodynamical profile in Theorem 2.13 and of the stationary profile in Theorem 2.15. Section 3 is devoted to the proof of Theorem 2.12 while Sections 4 and 5 are dedicated, respectively, to the convergence of the hydrodynamical profile and of the stationary profile. Finally, in Section 6 we prove the uniqueness of all the weak solutions that we consider in this work.

2. Statement of Results

2.1. The Model

For $N \geq 2$ let $\Lambda_N = \{1, \dots, N-1\}$, which we refer to as the bulk. The boundary driven exclusion process with long jumps is a Markov process that we denote by $\{\eta(t)\}_{t\geq 0}$ with state space $\Omega_N := \{0,1\}^{\Lambda_N}$ and it is defined as follows. The configurations of the state space Ω_N are denoted by η, so that for $x \in \Lambda_N$, $\eta_x = 0$ means that the site x is vacant while $\eta_x = 1$ means that the site x is occupied. Fix $\gamma \in (1,2)$. Let $p : \mathbb{Z} \to [0,1]$ be a translation invariant transition probability defined by

$$p(z) = c_\gamma \frac{\mathbb{1}_{\{z \neq 0\}}}{|z|^{\gamma+1}}, \tag{2.1}$$

where $c_\gamma > 0$ is a normalizing constant. Since $\gamma \in (1,2)$, we know that p has infinite variance but finite mean.

We consider the process in contact with infinitely many stochastic reservoirs at the left and right of Λ_N. We fix the parameters $\alpha, \beta \in (0,1)$, $\kappa > 0$ and $\theta \leq 0$. Particles can be injected into any site z of the bulk from: the left of 0 at rate $\alpha\kappa N^{-\theta} p(z)$ or from the right of N at rate $\beta\kappa N^{-\theta} p(z)$. Particles can be removed from any site of the bulk to: the left of 0 at rate $(1-\alpha)\kappa N^{-\theta} p(z)$ and to the right of N at rate $(1-\beta)\kappa N^{-\theta} p(z)$. To properly describe the dynamics, at each pair of sites of the bulk $\{x, y\}$ we associate a Poisson process of intensity one and Poisson processes associated with different bonds are independent. Whenever a clock associated with a bond $\{x, y\}$ rings, the values of η_x and η_y are exchanged with rate $p(y-x)/2$. At the boundary the dynamics is described as follows. To each pair of sites $\{x, y\}$ with $x \in \Lambda_N$ and $y \leq 0$ (resp. $y \geq N$) we associate a Poisson process of intensity one and of them are independent. If the clock associated with the bond $\{x, y\}$ rings, the value of η_x changes to $1-\eta_x$ with rate $\kappa N^{-\theta} p(x-y)\,[(1-\alpha)\eta_x + \alpha(1-\eta_x)]$ (resp. $\kappa N^{-\theta} p(x-y)\,[(1-\beta)\eta_x + \beta(1-\eta_x)]$). The dynamics is illustrated in the Fig. 1.

The process is characterized by its infinitesimal generator

$$L_N = L_N^0 + \kappa N^{-\theta} L_N^\ell + \kappa N^{-\theta} L_N^r, \tag{2.2}$$

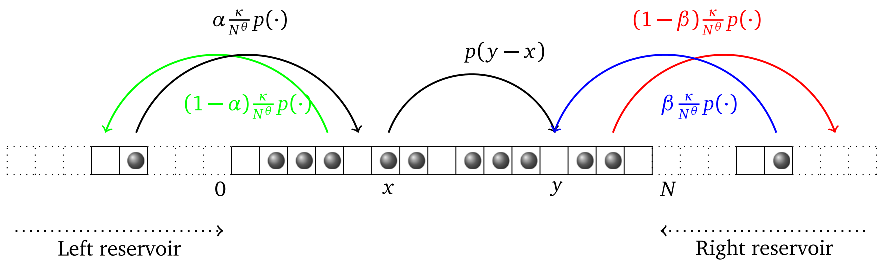

Fig. 1. Symmetric exclusion with long jumps given by the transtion probability $p(\cdot)$ and in contact with infinitely many reservoris with density α at the left of the system and β. at the right of the system

which acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$
\begin{aligned}
(L_N^0 f)(\eta) &= \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\sigma^{x,y}\eta) - f(\eta)], \\
(L_N^\ell f)(\eta) &= \sum_{\substack{x \in \Lambda_N \\ y \le 0}} p(x-y) c_x(\eta;\alpha)[f(\sigma^x\eta) - f(\eta)], \\
(L_N^r f)(\eta) &= \sum_{\substack{x \in \Lambda_N \\ y \ge N}} p(x-y) c_x(\eta;\beta)[f(\sigma^x\eta) - f(\eta)],
\end{aligned}
\tag{2.3}
$$

where

$$
(\sigma^{x,y}\eta)_z = \begin{cases} \eta_z, & \text{if } z \neq x,y, \\ \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y \end{cases}, \quad (\sigma^x\eta)_z = \begin{cases} \eta_z, & \text{if } z \neq x, \\ 1-\eta_x, & \text{if } z = x, \end{cases}
$$

and for a function $\varphi : [0,1] \to \mathbb{R}$ and for $x \in \Lambda_N$ we used the notation

$$
c_x(\eta; \varphi(\cdot)) := \left[\eta_x \left(1 - \varphi(\tfrac{x}{N})\right) + (1-\eta_x)\varphi(\tfrac{x}{N})\right]. \tag{2.4}
$$

We consider the Markov process speeded up in the subdiffusive time scale $t\Theta(N)$ and we use the notation $\eta_t^N := \eta(t\Theta(N))$, so that η_t^N has infinitesimal generator $\Theta(N)L_N$. Although η_t^N depends on α, β θ and κ, we shall omit these indexes in order to simplify notation.

2.2. Hydrodynamic Equations

From now and for the rest of this article we fix a finite time window $[0,T]$. To properly state the hydrodynamic limit, we need to introduce some notations and definitions, which we present as follows: first we abbreviate the Hilbert space $L^2([0,1], h(u)\mathrm{d}u)$ by L_h^2 and we denote its inner product by $\langle \cdot, \cdot \rangle_h$ and the corresponding norm by $\| \cdot \|_h$. When $h \equiv 1$ we simply write L^2, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. For an interval I in $\mathbb{R}$ and integers m and n, we denote by $C^{m,n}([0,T] \times I)$ the set of functions defined on $[0,T] \times I$ that are m times differentiable on the first variable and n times differentiable on the second variable, with continuous derivatives. We denote by $C_c^\infty(I)$ the set of all smooth real-valued functions defined in I with compact support included in I. The supremum norm is denoted by $\| \cdot \|_\infty$. We also consider the set $C_c^{1,\infty}([0,T] \times I)$ of functions $G \in C^{1,\infty}([0,T] \times I)$ such that $G(t,\cdot) \in C_c^\infty(I)$ for all $t \in [0,T]$. An index on a function will always denote a variable, not a derivative. For example, $G_t(u)$ means $G(t,u)$. The derivative of $G \in C^{m,n}([0,T] \times I)$ will be denoted by $\partial_t G$ (first variable) and $\partial_u G$ (second variable).

The fractional Laplacian $-(-\Delta)^{\gamma/2}$ of exponent $\gamma/2$ is defined on the set of functions $G : \mathbb{R} \to \mathbb{R}$ such that

$$
\int_{-\infty}^{\infty} \frac{|G(u)|}{(1+|u|)^{1+\gamma}} \mathrm{d}u < \infty \tag{2.5}
$$

by

$$- (-\Delta)^{\gamma/2} G\,(u) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbb{1}_{|u-v|\geq\varepsilon} \frac{G(v) - G(u)}{|u-v|^{1+\gamma}} \mathrm{d}v, \tag{2.6}$$

provided the limit exists (which is the case, for example, if G is in the Schwartz space) and where c_γ is set in (2.1). Up to a multiplicative constant, $-(-\Delta)^{\gamma/2}$ is the generator of a γ-Lévy stable process.

We define the operator $\mathbb{L}$ by its action on functions $G \in C_c^\infty((0,1))$, by

$$\forall u \in (0,1), \quad (\mathbb{L}G)(u) = c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \nVdash_{|u-v|\geq\varepsilon} \frac{G(v) - G(u)}{|u-v|^{1+\gamma}} \mathrm{d}v.$$

The operator $\mathbb{L}$ is called the *regional fractional Laplacian* on $(0,1)$. The semi inner-product $\langle \cdot, \cdot \rangle_{\gamma/2}$ is defined on the set $C_c^\infty((0,1))$ by

$$\langle G, H \rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint_{[0,1]^2} \frac{(H(u) - H(v))(G(u) - G(v))}{|u-v|^{1+\gamma}} \mathrm{d}u\mathrm{d}v. \tag{2.7}$$

The corresponding semi-norm is denoted by $\| \cdot \|_{\gamma/2}$. Observe that for any $G, H \in C_c^\infty((0,1))$ we have that

$$\langle G, -\mathbb{L}H \rangle = \langle -\mathbb{L}G, H \rangle = \langle G, H \rangle_{\gamma/2}.$$

Recall (1.2). We introduced a family of operators indexed by κ and taking the form

$$\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1,$$

where V_1 was defined in 1.2. Acting on $C_c^\infty((0,1))$ these operators are symmetric and non-positive. For $\kappa = 1$, we recover the so-called restricted fractional Laplacian (see [23]):

$$\forall u \in (0,1), \quad -(-\Delta)^{\gamma/2} G\,(u) = (\mathbb{L}G)(u) - V_1(u)G(u) := (\mathbb{L}_1 G)(u), \tag{2.8}$$

while in the limit $\kappa \to 0$ we get the regional fractional Laplacian.

We rewrite $V_1(u) = r^-(u) + r^+(u)$ and $V_0(u) = \alpha r^-(u) + \beta r^+(u)$ where the functions $r^\pm : (0,1) \to (0,\infty)$ are defined by

$$r^-(u) = c_\gamma \gamma^{-1} u^{-\gamma}, \quad r^+(u) = c_\gamma \gamma^{-1} (1-u)^{-\gamma}. \tag{2.9}$$

Definition 2.1. The Sobolev space $\mathscr{H}^{\gamma/2} := \mathscr{H}^{\gamma/2}([0,1])$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$. This is a Hilbert space for the norm $\| \cdot \|_{\mathscr{H}^{\gamma/2}}$ defined by

$$\|g\|_{\mathscr{H}^{\gamma/2}}^2 := \|g\|^2 + \|g\|_{\gamma/2}^2.$$

Its elements coincide a.e. with continuous functions. The completion of $C_c^\infty((0,1))$ for this norm is denoted by $\mathscr{H}_0^{\gamma/2} := \mathscr{H}_0^{\gamma/2}([0,1])$. This is a Hilbert space whose elements coincide a.e. with continuous functions vanishing at 0 and 1. On $\mathscr{H}_0^{\gamma/2}$, the two norms $\|\cdot\|_{\mathscr{H}^{\gamma/2}}$ and $\|\cdot\|_{\gamma/2}$ are equivalent.

The space $L^2(0,T;\mathscr{H}^{\gamma/2})$ is the set of measurable functions $f:[0,T]\to\mathscr{H}^{\gamma/2}$ such that

$$\int_0^T \|f_t\|^2_{\mathscr{H}^{\gamma/2}}\mathrm{d}t<\infty.$$

The spaces $L^2(0,T;\mathscr{H}_0^{\gamma/2})$ and $L^2(0,T;L^2_h)$ are defined similarly.

We now extend the definition of the regional fractional Laplacian on $(0,1)$, which has been defined on $C^\infty((0,1))$, to the space $\mathscr{H}^{\gamma/2}$.

Definition 2.2. For $\rho\in\mathscr{H}^{\gamma/2}$ we define the distribution $\mathbb{L}\rho$ by

$$\langle\mathbb{L}\rho,G\rangle=\langle\rho,\mathbb{L}G\rangle,\quad G\in C_c^\infty((0,1)).$$

Let us check that $\mathbb{L}\rho$ is indeed a well defined distribution. Consider a sequence $\{G_n\}_{n\geq1}\in C_c^\infty((0,1))$ converging to 0 in the usual topology of the test functions. By the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [15]) we have for any $\rho\in\mathscr{H}^{\gamma/2}$ that $\langle\mathbb{L}\rho,G_n\rangle=\langle\rho,G_n\rangle_{\gamma/2}$. Now using the Cauchy–Schwarz's inequality and the mean value Theorem, we get that $\langle\mathbb{L}\rho,G_n\rangle$ is bounded from above by a constant times

$$\|\rho\|_{\gamma/2}\|G_n\|_{\gamma/2}\lesssim\|\rho\|_{\gamma/2}\|\partial_uG_n\|_\infty^2\iint_{[0,1]^2}|u-v|^{1-\gamma}\mathrm{d}u\mathrm{d}v$$

which goes to 0 as $n\to\infty$ since $\gamma\in(1,2)$. Therefore $\mathbb{L}\rho$ is a well defined distribution.

Above (and hereinafter) we write $f(u)\lesssim g(u)$ if there exists a constant C independent of u such that $f(u)\leq Cg(u)$ for every u. We will also write $f(u)=O(g(u))$ if the condition $|f(u)|\lesssim|g(u)|$ is satisfied. Sometimes, in order to stress the dependence of a constant C on some parameter a, we write $C(a)$.

2.3. Hydrodynamic Equations

Now, for the following definitions recall the definition of $\mathbb{L}_\kappa$ given in (1.2) and V_0 from (1.3).

Definition 2.3. Let $\hat\kappa\geq0$ be some parameter and let $g:[0,1]\to[0,1]$ be a measurable function. We say that $\rho^{\hat\kappa}:[0,T]\times[0,1]\to[0,1]$ is a weak solution of the non-homogeneous regional fractional reaction–diffusion equation with Dirichlet boundary conditions given by

$$\begin{cases}\partial_t\rho_t^{\hat\kappa}(u)=\mathbb{L}_{\hat\kappa}\rho_t^{\hat\kappa}(u)+\hat\kappa V_0(u), & (t,u)\in[0,T]\times(0,1),\\ \rho_t^{\hat\kappa}(0)=\alpha,\quad \rho_t^{\hat\kappa}(1)=\beta, & t\in[0,T],\\ \rho_0^{\hat\kappa}(u)=g(u), & u\in(0,1),\end{cases}\tag{2.10}$$

if :

(i) $\rho^{\hat\kappa} \in L^2(0,T;\mathscr{H}^{\gamma/2})$.
(ii) $\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t^{\hat\kappa}(u))^2}{u^\gamma} + \frac{(\beta-\rho_t^{\hat\kappa}(u))^2}{(1-u)^\gamma} \right\} \mathrm{d}u\, \mathrm{d}t < \infty$ for $\hat\kappa > 0$; $\rho_t^{\hat\kappa}(0) = \alpha, \rho_t^{\hat\kappa}(1) = \beta$ for almost every $t \in [0,T]$, for $\hat\kappa = 0$.
(iii) For all $t \in [0,T]$ and all functions $G \in C_c^{1,\infty}([0,T]\times(0,1))$ we have that

$$F_{Dir}(t,\rho^{\hat\kappa},G,g) := \left\langle \rho_t^{\hat\kappa}, G_t \right\rangle - \langle g, G_0\rangle - \int_0^t \left\langle \rho_s^{\hat\kappa}, \left(\partial_s + \mathbb{L}_{\hat\kappa}\right) G_s \right\rangle \mathrm{d}s - \hat\kappa \int_0^t \langle G_s, V_0\rangle\, \mathrm{d}s = 0. \tag{2.11}$$

Remark 2.4. Note that item (ii) is different for $\hat\kappa > 0$ and $\hat\kappa = 0$. We can see that the condition for $\hat\kappa = 0$ is weaker than the condition for $\hat\kappa > 0$. In fact, item (i) and item (ii) for $\hat\kappa > 0$ of the previous definition imply that $\rho_t^{\hat\kappa}(0) = \alpha$ and $\rho_t^{\hat\kappa}(1) = \beta$, for almost every t in $[0,T]$. Indeed, first note that by item (i) we know that ρ_t is $\frac{\gamma-1}{2}$-Hölder for almost every t in $[0,T]$ (see Theorem 8.2 of [13]). Then, we note that

$$\int_0^T \frac{(\rho_t^{\hat\kappa}(0)-\alpha)^2}{\gamma-1}\mathrm{d}t = \int_0^T \lim_{\varepsilon\to 0} \varepsilon^{\gamma-1} \int_\varepsilon^1 \frac{(\rho_t^{\hat\kappa}(0)-\alpha)^2}{u^\gamma}\mathrm{d}u\mathrm{d}t.$$

By summing and subtracting $\rho_t^{\hat\kappa}(u)$ inside the square in the expression on the right hand side in the previous equality and using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ we get that the right hand side of the previous equality is bounded from above by

$$2\int_0^T \lim_{\varepsilon\to 0} \varepsilon^{\gamma-1} \int_\varepsilon^1 \frac{(\rho_t^{\hat\kappa}(0)-\rho_t^{\hat\kappa}(u))^2}{u^\gamma}\mathrm{d}u\mathrm{d}t + 2\int_0^T \lim_{\varepsilon\to 0} \varepsilon^{\gamma-1} \int_\varepsilon^1 \frac{(\rho_t^{\hat\kappa}(u)-\alpha)^2}{u^\gamma}\mathrm{d}u\mathrm{d}t.$$

Since ρ_t is $\frac{\gamma-1}{2}$-Hölder for almost every t in $[0,T]$ the first term in the previous expression vanishes. Now, the term on the right hand side in the previous expression is bounded from above by

$$2\lim_{\varepsilon\to 0} \varepsilon^{\gamma-1} \int_0^T \int_0^1 \frac{(\rho_t^{\hat\kappa}(u)-\alpha)^2}{u^\gamma}\mathrm{d}u\mathrm{d}t,$$

which vanishes as a consequence of item (ii). Thus, we have that

$$\int_0^T \frac{(\rho_t^{\hat\kappa}(0)-\alpha)^2}{\gamma-1}\mathrm{d}t = 0,$$

whence we get that $\rho_t^{\hat\kappa}(0) = \alpha$ for almost every t in $[0,T]$. Showing that $\rho_t^{\hat\kappa}(1) = \beta$ for almost every t in $[0,T]$ is completely analogous.

Moreover, the existence and uniqueness of a weak solution to the equation above, for $\hat\kappa > 0$ does not require the strong form of (ii). Nevertheless, in order to prove Theorem 2.13 we need to impose that condition.

Remark 2.5. Observe that in the case $\hat{\kappa} = 1$, since $\mathbb{L}_1 = -(-\Delta)^{\gamma/2}$ we obtain in Definition 2.3 the fractional heat equation with reaction and Dirichlet boundary conditions, i.e.

$$\begin{cases} \partial_t \rho_t^1(u) = \mathbb{L}_1 \rho_t^1(u) + V_0(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t^1(0) = \alpha, \quad \rho_t^1(1) = \beta, & t \in [0,T], \\ \rho_0^1(u) = g(u), \quad u \in (0,1), \end{cases}$$

by (2.8) and (1.2) the notion of item (iii) is reduced to

$$F_{Dir}(t, \rho^1, G, g) := \left\langle \rho_t^1, G_t \right\rangle - \langle g, G_0 \rangle - \int_0^t \left\langle \rho_s^1, \left(\partial_s - (-\Delta)^{\gamma/2}\right) G_s \right\rangle \mathrm{d}s \\ - \int_0^t \langle G_s, V_0 \rangle \, \mathrm{d}s = 0$$

for all $t \in [0,T]$ and all functions $G \in C_c^{1,\infty}([0,T] \times (0,1))$.

Definition 2.6. Let $\hat{\kappa} > 0$ be some parameter and let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the non-homogeneous reaction equation with Dirichlet boundary conditions given by

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = -\hat{\kappa} \rho_t^{\hat{\kappa}}(u) V_1(u) + \hat{\kappa} V_0(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t^{\hat{\kappa}}(0) = \alpha, \quad \rho_t^{\hat{\kappa}}(1) = \beta, & t \in [0,T], \\ \rho_0^{\hat{\kappa}}(u) = g(u), \quad u \in (0,1), \end{cases} \tag{2.12}$$

if:

(i) $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \rho_t^{\hat{\kappa}}(u))^2}{(1-u)^\gamma} \right\} \mathrm{d}u \, \mathrm{d}t < \infty$.

(ii) For all $t \in [0,T]$ and all functions $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have

$$F_{Reac}(t, \rho^{\hat{\kappa}}, G, g) := \left\langle \rho_t^{\hat{\kappa}}, G_t \right\rangle - \langle g, G_0 \rangle - \int_0^t \left\langle \rho_s^{\hat{\kappa}}, \partial_s G_s \right\rangle \mathrm{d}s \\ + \int_0^t \left\langle \rho_s^{\hat{\kappa}}, G_s \right\rangle_{V_1} \mathrm{d}s - \hat{\kappa} \int_0^t \langle G_s, V_0 \rangle \, \mathrm{d}s = 0. \tag{2.13}$$

Remark 2.7. Note that the explicit solution of (2.12) is given by

$$\bar{\rho}^\infty(u) + (g(u) - \bar{\rho}^\infty(u)) e^{-t\hat{\kappa} V_1(u)},$$

where $\bar{\rho}^\infty(u) = \dfrac{V_0(u)}{V_1(u)}$. As we will see, the function $\bar{\rho}^\infty$ plays an important role in the proof of our main results, namely, Theorems 2.13 and 2.15.

Lemma 2.8. *The weak solutions of* (2.10) *and* (2.12) *are unique.*

Aiming to concentrate on the main facts, the proof of previous lemma is postponed to Section 6.

Definition 2.9. Let $\hat{\kappa} \geq 0$ be some parameter. We say that $\bar{\rho}^{\hat{\kappa}} : [0, 1] \to [0, 1]$ is a weak solution of the stationary regional fractional reaction–diffusion equation with non-homogeneous Dirichlet boundary conditions given by

$$\begin{cases} \mathbb{L}_{\hat{\kappa}} \bar{\rho}^{\hat{\kappa}}(u) + \hat{\kappa} V_0(u) = 0, \quad u \in (0, 1), \\ \bar{\rho}^{\hat{\kappa}}(0) = \alpha, \quad \bar{\rho}^{\hat{\kappa}}(1) = \beta, \end{cases} \tag{2.14}$$

if:

(i) $\bar{\rho}^{\hat{\kappa}} \in \mathscr{H}^{\gamma/2}$.
(ii) $\int_0^1 \left\{ \frac{\left(\alpha - \bar{\rho}^{\hat{\kappa}}(u)\right)^2}{u^\gamma} + \frac{\left(\beta - \bar{\rho}^{\hat{\kappa}}(u)\right)^2}{u^\gamma} \right\} \mathrm{d}u < \infty$ if $\hat{\kappa} > 0$ and $\bar{\rho}^{\hat{\kappa}}(0) = \alpha$, $\bar{\rho}^{\hat{\kappa}}(1) = \beta$ if $\hat{\kappa} = 0$.
(iii) For any function $G \in C_c^\infty((0, 1))$ we have

$$\bar{F}_{Dir}(\bar{\rho}^{\hat{\kappa}}, G) := \left\langle \bar{\rho}^{\hat{\kappa}}, \mathbb{L}_{\hat{\kappa}} G \right\rangle + \hat{\kappa} \left\langle G, V_0 \right\rangle = 0.$$

Remark 2.10. We observe that $\bar{\rho}^0$ is a weak harmonic function for $\mathbb{L}$ and the interior regularity of this solution is studied in [21], but the regularity at the boundary is unknown.

In Section 6 we will prove the following lemma.

Lemma 2.11. *There exists a unique weak solution of* (2.14).

2.4. Statement of Results

First we want to state the hydrodynamic limit of the process $\{\eta_t^N\}_{t \geq 0}$ with state space Ω_N and with infinitesimal generator $\Theta(N) L_N$ defined in (2.2).

Let $\mathscr{M}^+$ be the space of positive measures on $[0, 1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$ we define the empirical measure $\pi^N(\eta, \mathrm{d}u) := \pi^{N,\kappa}(\eta, \mathrm{d}u)$ in Ω_N by

$$\pi^N(\eta, \mathrm{d}u) = \frac{1}{N - 1} \sum_{x \in \Lambda_N} \eta_x \delta_{\frac{x}{N}}(\mathrm{d}u), \tag{2.15}$$

where δ_a is a Dirac mass at $a \in [0, 1]$ and $\pi_t^N(\eta, \mathrm{d}u) := \pi^N(\eta_t^N, \mathrm{d}u)$.

Let $g : [0, 1] \to [0, 1]$ be a measurable function. We say that a sequence of probability measures $\{\mu_N\}_{N \geq 1}$ in Ω_N is associated to the profile g if for any continuous function $G : [0, 1] \to \mathbb{R}$ and every $\delta > 0$

$$\lim_{N \to \infty} \mu_N \left(\eta \in \Omega_N : \left| \frac{1}{N} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) \eta_x - \int_0^1 G(u) g(u) \mathrm{d}u \right| > \delta \right) = 0.$$

We denote by $\mathbb{P}_{\mu_N}$ the probability measure in the Skorohod space $\mathscr{D}([0, T], \Omega_N)$ induced by the Markov process η_t^N and the initial measure μ_N in Ω_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Let $\{\mathbb{Q}_N\}_{N \geq 1}$ be the sequence of probability measures on the Skorohod space $\mathscr{D}([0, T], \mathscr{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t \geq 0}$ and by $\mathbb{P}_{\mu_N}$.

At this point we are ready to state the hydrodynamic limit of the process η_t^N.

Theorem 2.12. (Hydrodynamic limit) *Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to g. Then, for any $0 \leq t \leq T$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N) : \left|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\eta_x(t\Theta(N)) - \int_0^1 G(u)\rho_t^{\hat{\kappa}}(u)\mathrm{d}u\right| > \delta\right) = 0,$$

where the time scale is given by $\Theta(N) = N^{\gamma+\theta}$ and $\rho_t^{\hat{\kappa}}$ is the unique weak solution of:

- (2.12) *with $\hat{\kappa} = \kappa$, if $\theta < 0$;*
- (2.10) *with $\hat{\kappa} = \kappa$, if $\theta = 0$.*

Once the hydrodynamic limit is obtained, we would like to know how the weak solution ρ_t^κ and the stationary solution $\bar{\rho}^\kappa$ behave as κ goes to 0 or ∞ and this is the purpose of Theorems 2.13 and 2.15 stated below. This limiting profile will give us an idea of what to expect at the hydrodynamics level when we consider our microscopic dynamics in contact with reservoirs whose strength is regulated by κ/N^θ and when $\theta > 0$ as in [4]. As mentioned in the introduction we do not analyze the system in this regime but we conjecture that for small positive values of $\theta > 0$ (that corresponds to slow reservoirs) the hydrodynamic limit should be given by the weak solution of (2.10) with $\hat{\kappa} = 0$.

Theorem 2.13. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Further, let ρ^κ be the weak solution of (2.10) with $\hat{\kappa} = \kappa$ and with initial condition ρ_0 which is independent of κ and let $\hat{\rho}_t^\kappa := \rho_{t/\kappa}^\kappa$, for all $t \in [0,T]$. Then*

(i) *ρ^κ converges strongly to ρ^0 in $L^2(0,T;\mathscr{H}^{\gamma/2})$ as κ goes to 0, where ρ^0 is the weak solution of (2.10) with $\hat{\kappa} = 0$ and initial condition ρ_0.*
(ii) *If $\rho_0 - \bar{\rho}^\infty \in \mathscr{H}^{\gamma/2}$ then $\hat{\rho}^\kappa$ converges strongly to ρ^∞ in $L^2(0,T;L^2_{V_1})$ as κ goes to ∞, where ρ^∞ is the weak solution of (2.12) with $\hat{\kappa} = 1$ and initial condition ρ_0.*

Remark 2.14. The convergence in Theorem 2.13 is also true in $L^2(0,T;L^2)$. In fact, we will see that a crucial step in the proof of the theorem is to show that ρ^κ converges strongly in $L^2(0,T;L^2)$. Convergence in i) is also true in $L^2(0,T;L^2_{V_1})$ and it is a consequence of the fractional Hardy's inequality (see e.g. [12]).

Theorem 2.15. *Let $\bar{\rho}^\kappa$ be the weak solution of (2.14). Then,*

(i) *$\bar{\rho}^\kappa$ converges strongly to $\bar{\rho}^0$ in $\mathscr{H}^{\gamma/2}$ as κ goes to 0, where $\bar{\rho}^0$ is the weak solution of (2.14) with $\kappa = 0$.*
(ii) *$\bar{\rho}^\kappa$ converges strongly to $\bar{\rho}^\infty$ in $L^2_{V_1}$ as κ goes to ∞, where $\bar{\rho}^\infty$ is given in Remark 2.7.*

3. Proof of Theorem 2.16: Hydrodynamic Limit

The proof of this theorem follows the usual approach of convergence in distribution of stochastic processes: we prove tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ and then we prove uniqueness of the limiting point, which we denote by $\mathbb{Q}$. These two results combined give the convergence of $\{\mathbb{Q}_N\}_{N\geq 1}$ to $\mathbb{Q}$, as $N \to \infty$. In order to characterize the limiting point $\mathbb{Q}$, we prove that all limiting points of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure and whose density ρ_t^κ is a weak solution of the hydrodynamic equation as given in Definition 2.3. From the uniqueness of the weak solutions of this equation, namely Lemma 2.11, we conclude that $\{\mathbb{Q}_N\}_{N\geq 1}$ has a unique limit point $\mathbb{Q}$.

First, in the following subsection we explain how the item (iii) in Definition 2.3 appears. In Section 3.2 we prove that $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight, then in Section 3.3 we obtain energy estimates which are crucial to ensure the uniqueness of the limiting point. We conclude this section with the characterization of the limiting point (in Section 3.4).

3.1. Heuristics for the Hydrodynamic Equations

In order to make the presentation simple, let us fix a function $G : [0, 1] \to \mathbb{R}$ which does not depend on time and has compact support included in $(0, 1)$.

By Dynkin's formula (see Lemma A.5.1 in [16]) we have that

$$M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \Theta(N) L_N \langle \pi_s^N, G\rangle \mathrm{d}s \tag{3.1}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where $\mathscr{F}_t := \sigma(\{\eta(s)\}_{s\leq t})$ for all $t \in [0, T]$.

Above, for an integrable function $G : [0, 1] \to \mathbb{R}$, we used the notation $\langle \pi_t^N, G\rangle$ to represent the integral of G with respect the measure π_t^N:

$$\left\langle \pi_t^N, G\right\rangle = \frac{1}{N-1} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \eta_x(t\Theta(N)).$$

In the previous expression, we are using a measure π_t^N and a function G, therefore, this notation should not be mistaken with the one used for the inner product in L^2. Note that $L_N \eta_x$ is equal to

$$\sum_{y\in\Lambda_N} p(x-y)[\eta_y - \eta_x] + \frac{\kappa}{N^\theta} \sum_{y\leq 0} p(x-y)[\alpha - \eta_x]$$
$$+ \frac{\kappa}{N^\theta} \sum_{y\geq N} p(x-y)[\beta - \eta_x].$$

Therefore, a simple computation shows that

$$\Theta(N)L_N(\langle\pi^N, G\rangle) = \frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}(\mathscr{L}_N G)(\tfrac{x}{N})\eta_x + \frac{\kappa\Theta(N)}{(N-1)N^\theta} \sum_{x\in\Lambda_N} G(\tfrac{x}{N})\left(r_N^-(\tfrac{x}{N})(\alpha-\eta_x) + r_N^+(\tfrac{x}{N})(\beta-\eta_x)\right), \tag{3.2}$$

where we denote by $\mathscr{L}_N G$ the continuous function on $[0, 1]$ which is defined as the linear interpolation of the function

$$(\mathscr{L}_N G)(\tfrac{x}{N}) = \sum_{y\in\Lambda_N} p(y-x)\left[G(\tfrac{y}{N}) - G(\tfrac{x}{N})\right] \tag{3.3}$$

for all $x \in \Lambda_N$ with $(\mathscr{L}_N G)(0) = (\mathscr{L}_N G)(1) = 0$. We also define the functions $r_N^\pm : [0, 1] \to \mathbb{R}$ as the linear interpolation of the function

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y) \tag{3.4}$$

for all $x \in \Lambda_N$ with $r_N^\pm(0) = r_N^\pm(\frac{1}{N})$ and $r_N^\pm(1) = r_N^\pm(\frac{N-1}{N})$. By Lemma 3.3 in [3] we have that

$$\lim_{N\to\infty} N^\gamma(r_N^-)(u) = r^-(u), \quad \lim_{N\to\infty} N^\gamma(r_N^+)(u) = r^+(u) \tag{3.5}$$

uniformly in $[a, 1-a]$ for $a \in (0, 1)$ and we also can deduce from that lemma that

$$\lim_{N\to\infty} N^\gamma(\mathscr{L}_N G)(u) = (\mathbb{L}G)(u) \tag{3.6}$$

uniformly in $[a, 1-a]$, for all functions G with compact support included in $[a, 1-a]$.

Now, we are going to analyse all the terms in (3.2) for $\theta \leq 0$. Thus, we will be able to see how the different boundary conditions appear on the hydrodynamic equations given in Section 2.3 from the underlying particle system.

3.1.1. The Case $\theta < 0$ In this regime we take $\Theta(N) = N^{\gamma+\theta}$ and a function $G \in C_c^\infty(0, 1)$. By using (3.6) we have that the first term on the right hand side of (3.2) vanishes since $\theta < 0$. Now, the second term on the right hand side in (3.2) is equal to $\kappa\langle\alpha - \pi_t^N, N^\gamma G r_N^-\rangle + \kappa\langle\beta - \pi_t^N, N^\gamma G r_N^+\rangle$. By (3.5) the previous expression converges, as N goes to ∞, to

$$\kappa\int_0^1(\alpha - \rho_t^\kappa(u))G(u)r^-(u)\mathrm{d}u + \kappa\int_0^1(\beta - \rho_t^\kappa(u))G(u)r^+(u)\mathrm{d}u$$
$$= -\kappa\int_0^1 \rho_t^\kappa(u)G(u)V_1(u)\mathrm{d}u + \kappa\int_0^1 G(u)V_0(u)\mathrm{d}u.$$

3.1.2. The Case $\theta = 0$ In this regime we take N^γ and a function $G \in C_c^\infty(0,1)$. The first term on the right hand side in (3.2) can be replaced, thanks to (3.6) by

$$\langle \pi_t^N, \mathbb{L}G\rangle \to \int_0^1 (\mathbb{L}G)(u)\rho_t^\kappa(u)\mathrm{d}u,$$

as N goes to ∞. Similarly, the second term on the right hand side of (3.2) is equal to $\kappa\langle \alpha - \pi_t^N, N^\gamma G r_N^-\rangle + \kappa\langle \beta - \pi_t^N, N^\gamma G r_N^+\rangle$ which converges, as N goes to ∞, to

$$\begin{aligned}&\kappa\int_0^1 (\alpha - \rho_t^\kappa(u))G(u)r^-(u)\mathrm{d}u + \kappa\int_0^1 (\beta - \rho_t^\kappa(u))G(u)r^+(u)\mathrm{d}u\\ &= -\kappa\int_0^1 \rho_t^\kappa(u)G(u)V_1(u)\mathrm{d}u + \kappa\int_0^1 G(u)V_0(u)\mathrm{d}u.\end{aligned}$$

This intuitive argument is rigorously proved in Section 3.4.

3.2. Tightness

In this subsection we prove that the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight. We use the usual approach (see, for example, Propositions 4.1.6 and 4.1.7 in [16]), which says that is enough to show that, for all $\varepsilon > 0$

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\leq\delta}\mathbb{P}_{\mu_N}\Big[\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N)\\ : \Big|\langle \pi_{\tau+\bar\tau}^N, G\rangle - \langle \pi_\tau^N, G\rangle\Big| > \varepsilon\Big] = 0,\end{aligned} \tag{3.7}$$

for any function G belonging to $C([0,1])$. Above $\mathscr{T}_T$ is the set of stopping times bounded by T and we implicitly assume that all the stopping times are bounded by T, thus, $\tau + \bar\tau$ should be read as $(\tau+\bar\tau)\wedge T$. Indeed, we prove below that (3.7) is true for any function G in $C_c^2((0,1))$, by using an L^1 approximation procedure(a similar argument as done in [4]), we can extend this class of functions to functions $G \in C([0,1])$.

Proposition 3.1. *The sequence of measures $\{\mathbb{Q}_N\}_{N\geq1}$ is tight with respect to the Skorohod topology of $\mathscr{D}([0,T],\mathscr{M}^+)$.*

Proof. Note that, we are going to prove (3.7) for functions G in $C_c^2((0,1))$. Recall from (3.1) that $M_t^N(G)$ is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. In order to prove (3.7) it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\leq\delta}\mathbb{E}_{\mu_N}\left[\left|\int_\tau^{\tau+\bar\tau}\Theta(N)L_N\langle \pi_s^N, G\rangle\mathrm{d}s\right|\right] = 0 \tag{3.8}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\leq\delta}\mathbb{E}_{\mu_N}\left[\Big(M_\tau^N(G) - M_{\tau+\bar\tau}^N(G)\Big)^2\right] = 0. \tag{3.9}$$

By using (3.5), (3.6) and the fact that $G \in C_c^2((0,1))$ we can bound the expression in (3.2) by a constant. By using the fact that $|\eta_x^N(s)| \le 1$ and

$$\sum_{x \geq 1} \left(r_N^-(\tfrac{x}{N}) + r_N^+(\tfrac{x}{N}) \right) < \infty \tag{3.10}$$

(since $\gamma > 1$), we can bound from above the second term at the right hand side in (3.2) by a constant times $\Theta(N) N^{-1-\theta}$. Considering the different values of θ we see that such term is bounded from above by a constant. Then we have that

$$|\Theta(N) L_N(\langle \pi_s^N, G\rangle)| \lesssim 1 \tag{3.11}$$

for any $s \le T$, which trivially implies (3.8).

In order to prove (3.9), by Dynkin's formula (see Appendix 1 in [16]) we know that

$$\left(M_t^N(G)\right)^2 - \int_0^t \Theta(N) \left[L_N \langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle L_N \langle \pi_s^N, G\rangle \right] \mathrm{d}s$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t \geq 0}$. By Lemma A.1 we get that the term inside the time integral in the previous expression is equal to

$$\begin{aligned} &\frac{\Theta(N)}{(N-1)^2} \sum_{x<y \in \Lambda_N} \left(G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y}{N}\right) \right)^2 p(x-y) (\eta_y(s\Theta(N)) - \eta_x(s\Theta(N)))^2 \\ &+ \frac{\kappa \Theta(N)}{(N-1)^2 N^\theta} \sum_{x \in \Lambda_N} \left(G\left(\tfrac{x}{N}\right)\right)^2 (1 - 2\eta_x(s\Theta(N))) r_N^-(\tfrac{x}{N}) (\alpha - \eta_x(s\Theta(N))) \\ &+ \frac{\kappa \Theta(N)}{(N-1)^2 N^\theta} \sum_{x \in \Lambda_N} \left(G\left(\tfrac{x}{N}\right)\right)^2 (1 - 2\eta_x(s\Theta(N))) r_N^+(\tfrac{x}{N}) (\beta - \eta_x(s\Theta(N))). \end{aligned} \tag{3.12}$$

Since the first derivative of G is bounded it is easy to see that the absolute value of (3.12) is bounded from above by a constant times

$$\begin{aligned} &\frac{\Theta(N)}{(N-1)^4} \sum_{x,y \in \Lambda_N} (x-y)^2 p(x-y) \\ &+ \frac{\kappa \Theta(N)}{(N-1)^2 N^\theta} \sum_{x \in \Lambda_N} \left(G\left(\tfrac{x}{N}\right)\right)^2 \left(r_N^-(\tfrac{x}{N}) + r_N^+(\tfrac{x}{N}) \right). \end{aligned} \tag{3.13}$$

Note that $(x-y)^2 p(x-y) \lesssim 1$ because $\gamma > 1$, so that

$$\frac{\Theta(N)}{(N-1)^4} \sum_{x,y \in \Lambda_N} (x-y)^2 p(x-y) \lesssim \Theta(N) N^{-2} = \mathscr{O}(N^{\gamma-2}).$$

By (3.10), the remaining terms in (3.13) are $\mathscr{O}(\Theta(N) N^{-\theta-2})$ so that (3.13) is $\mathscr{O}(N^{\gamma-2})$.

Thus, since τ is a stopping time and $\gamma < 2$, we have that

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar{\tau}\leq\delta}\mathbb{E}_{\mu^N}\left[\left(M_\tau^{N,G}-M_{\tau+\bar{\tau}}^{N,G}\right)^2\right]\\&=\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar{\tau}\leq\delta}\mathbb{E}_{\mu^N}\\&\quad\left[\int_\tau^{\tau+\bar{\tau}}\Theta(N)\left[L_N\langle\pi_s^N,G\rangle^2-2\langle\pi_s^N,G\rangle L_N\langle\pi_s^N,G\rangle\right]\mathrm{d}s\right]\\&=0.\end{aligned}$$

Therefore, we have proved (3.7) for functions G in $C_c^2((0,1))$ and as we have said in the beginning of the subsection this is enough to conclude tightness. □

3.3. Energy Estimate

We prove in this subsection that any limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ is concentrated on trajectories $\pi_t^\kappa(u)\mathrm{d}u$ with finite energy, i.e., π^κ belongs to $L^2(0,T;\mathscr{H}^{\gamma/2})$. Moreover, we prove that π_t^κ satisfies item (ii) in Definition 2.3. The latter is the content of Theorem 3.2 stated below. Fix a limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ and assume, without of loss of generality, that the sequence $\mathbb{Q}_N$ converges to $\mathbb{Q}$ as N goes to ∞.

Theorem 3.2. *The probability measure $\mathbb{Q}$ is concentrated on trajectories of measures of the form $\pi_t^\kappa(u)du$, such that for any interval $I\subset[0,T]$ the density π^κ satisfies*

(i) $\int_I\|\pi_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\lesssim|I|(\kappa+1)$, *if* $\theta=0$.

(ii) $\displaystyle\int_I\int_0^1\left\{\frac{(\alpha-\pi_t^\kappa(u))^2}{u^\gamma}+\frac{(\beta-\pi_t^\kappa(u))^2}{(1-u)^\gamma}\right\}\mathrm{d}u\,\mathrm{d}t\lesssim|I|\frac{\kappa 1}{\kappa}$, *if* $\theta\leq 0$.

Remark 3.3. It follows from item (i) of the previous and from Theorem 8.2 of [13] that π_t^κ is, $\mathbb{P}$ almost surely, $\frac{\gamma-1}{2}$-Hölder for all $t\in I$.

By taking $I=[0,T]$ in item (i) of Theorem 3.2 we can see that $\pi^\kappa\in L^2(0,T;\mathscr{H}^{\gamma/2})$. Moreover, from item (ii) of Theorem 3.2, we claim that

$$\int_I\|\pi_t^\kappa-\bar{\rho}^\infty\|_{V_1}^2\mathrm{d}t\lesssim|I|\frac{\kappa+1}{\kappa},\tag{3.14}$$

where $\bar{\rho}^\infty$ is given in Remark 2.7. Note that

$$\begin{aligned}\int_I\|\pi_t^\kappa-\bar{\rho}^\infty\|_{V_1}^2\mathrm{d}t=c_\gamma\gamma^{-1}\int_I\int_0^1&\left\{\frac{(\pi_t^\kappa(u)-\bar{\rho}^\infty(u))^2}{u^\gamma}\right.\\&\left.+\frac{(\pi_t^\kappa(u)-\bar{\rho}^\infty(u))^2}{(1-u)^\gamma}\right\}\mathrm{d}u\mathrm{d}t.\end{aligned}\tag{3.15}$$

By summing and subtracting α inside the first square in the expression on the right hand side in (3.15), β in the second one and using the fact that $(a+b)^2 \le 2(a^2+b^2)$ we get that (3.15) is bounded from above by

$$\begin{aligned} &2c_\gamma \gamma^{-1} \int_I \int_0^1 \left\{ \frac{(\pi_t^\kappa(u)-\alpha)^2}{u^\gamma} + \frac{(\pi_t^\kappa(u)-\beta)^2}{(1-u)^\gamma} \right\} \mathrm{d}u\mathrm{d}t \\ &\quad + 2c_\gamma \gamma^{-1} \int_I \int_0^1 \left\{ \frac{(\alpha-\bar\rho^\infty(u))^2}{u^\gamma} + \frac{(\beta-\bar\rho^\infty(u))^2}{(1-u)^\gamma} \right\} \mathrm{d}u\mathrm{d}t. \end{aligned} \tag{3.16}$$

Now, by using item ii) of Theorem 3.2 we have that the first term in the previous expression is bounded by constant times $|I|\dfrac{\kappa+1}{\kappa}$. Finally, using the definition of $\bar\rho^\infty$ (see Remark 2.7) the second term in (3.16) is equal to

$$2c_\gamma \gamma^{-1}(\beta-\alpha)^2|I| \int_0^1 (u^\gamma + (1-u)^\gamma)^{-1}\mathrm{d}u \lesssim 1.$$

Before we prove Theorem 3.2, we establish some estimates on the Dirichlet form which are needed in due course.

3.3.1. Estimates on the Dirichlet Form Let $h : [0,1] \to [0,1]$ be a function such that $\alpha \le h(u) \le \beta$, for all $u \in [0,1]$, and assume that $h(0)=\alpha$ and $h(1)=\beta$. Let ν_h^N be the inhomogeneous Bernoulli product measure on Ω_N with marginals given by

$$\nu_h^N\{\eta : \eta_x = 1\} = h\left(\tfrac{x}{N}\right).$$

We denote by $H_N(\mu|\nu_h^N)$ the relative entropy of a probability measure μ on Ω_N with respect to the probability measure ν_h^N. It is easy to prove the existence of a constant C_0, such that

$$H_N(\mu_N|\nu_h^N) \le C_0 N \tag{3.17}$$

(see for example [4]). We remark here that the restriction $\alpha \ne 0$ and $\beta \ne 1$ comes from last estimate since the constant C_0 given above is given by $C_0 = -\log(\alpha \wedge (1-\beta))$. On the other hand, for a probability measure μ on Ω_N and a density function $f : \Omega_N \to [0,\infty)$ with respect to μ we introduce

$$D_N^0(\sqrt{f},\mu) := \frac{1}{2} \sum_{x,y\in\Lambda_N} p(y-x)\, I_{x,y}(\sqrt{f},\mu), \tag{3.18}$$

$$D_N^\ell(\sqrt{f},\mu) := \sum_{x\in\Lambda_N} \sum_{y\le 0} p(y-x)\, I_x^\alpha(\sqrt{f},\mu) = \sum_{x\in\Lambda_N} r_N^-\left(\tfrac{x}{N}\right) I_x^\alpha(\sqrt{f},\mu) \tag{3.19}$$

and $D_N^r(\sqrt{f},\mu)$ is the same as $D_N^\ell(\sqrt{f},\mu)$ but in $I_x^\alpha(\sqrt{f},\mu)$ the parameter α is replaced by β and r_N^- is replaced by r_N^+. Above, we used the following notation:

$$I_{x,y}(\sqrt{f},\mu) := \int \left(\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)}\right)^2 \mathrm{d}\mu,$$

$$I_x^\alpha(\sqrt{f},\mu) := \int c_x(\eta;\alpha)\left(\sqrt{f(\sigma^x\eta)}-\sqrt{f(\eta)}\right)^2 \mathrm{d}\mu,$$

where $c_x(\eta,\alpha)$ is given in (2.4) with $\varphi(\cdot)\equiv\alpha$; and I_x^β is the same as I_x^α when the parameter α is replaced by β.

Our goal is to express, for the measure ν_h^N, a relation between the Dirichlet form defined by $\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}$ and the quantity

$$D_N(\sqrt{f},\nu_h^N) := (D_N^0+\kappa N^{-\theta}D_N^\ell+\kappa N^{-\theta}D_N^r)(\sqrt{f},\nu_h^N).$$

More precisely, we have the following result:

Lemma 3.4. *For any positive constant B and any density function f with respect to ν_h^N, there exists a constant $C>0$ (independent of f and N) such that*

$$\begin{aligned}&\frac{\Theta(N)}{NB}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\\ &\quad\le -\frac{\Theta(N)}{4NB}D_N(\sqrt{f},\nu_h^N)+\frac{C\Theta(N)}{NB}\sum_{x,y\in\Lambda_N}p(y-x)\Big(h(\tfrac{x}{N})-h(\tfrac{y}{N})\Big)^2\\ &\qquad+\frac{C\kappa\Theta(N)}{N^{\theta+1}B}\sum_{x\in\Lambda_N}\left\{\Big(h(\tfrac{x}{N})-\alpha\Big)^2 r_N^-(\tfrac{x}{N})+\Big(h(\tfrac{x}{N})-\beta\Big)^2 r_N^+(\tfrac{x}{N})\right\}.\end{aligned}\tag{3.20}$$

The proof of this statement is similar to the one in Section 5 of [4] and thus it is omitted. Moreover, note that as a consequence of the previous lemma, for a function h such that $\alpha\le h(u)\le\beta$ and h Lipschitz we have that

$$\frac{\Theta(N)}{NB}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\le -\frac{\Theta(N)}{4NB}D_N(\sqrt{f},\nu_h^N)+\Theta(N)N^{-\gamma}\tfrac{C(\kappa N^{-\theta}+1)}{B}.\tag{3.21}$$

Lemma 3.5. *For any density f with respect to ν_h^N, any $x\in\Lambda_N$ and any positive constant A_x, we have that*

$$\left|\langle\eta_x-\alpha,f\rangle_{\nu_h^N}\right|\lesssim\frac{1}{4A_x}I_x^\alpha(\sqrt{f},\nu_h^N)+A_x+\left|h(\tfrac{x}{N})-\alpha\right|.$$

The same result holds if α is replaced by β.

The proof of Lemma 3.5 is omitted since is similar to the one of Lemma 5.5 in [4]. Note that in the case $\alpha\le h\le\beta$ and Lipschitz we get

$$\left|\langle\eta_x-\alpha,f\rangle_{\nu_h^N}\right|\lesssim\frac{1}{4A_x}I_x^\alpha(\sqrt{f},\nu_h^N)+A_x+\frac{x}{N}.$$

3.3.2. Proof of Theorem 3.2 **First item:** $\pi^\kappa \in L^2(0,T;\mathscr{H}^{\gamma/2})$ $\mathbb{Q}$-almost surely. Recall that in this case $\theta = 0$ and the system is speeded up in the sub-diffusive time scale $\Theta(N) = N^\gamma$. Let $\varepsilon > 0$ be a small real number. Let $F \in C_c^{0,\infty}(I \times [0,1]^2)$, where the I is a subinterval of $[0,T]$. Observe that by the entropy inequality

$$\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y(tN^\gamma)-\eta_x(tN^\gamma))\mathrm{d}t\right]$$

is bounded from above by

$$C_0 + \frac{1}{N}\log\int\left[e^{\mathbb{E}_\eta[|\int_I NG_N(t,\eta_{tN^\gamma})\,\mathrm{d}t|]}\right]\nu_h^N(\mathrm{d}\eta), \tag{3.22}$$

where

$$G_N(t,\eta) = N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y-\eta_x),$$

and by Jensen's inequality we can bound last expression from above by

$$C_0 + \frac{1}{N}\log\mathbb{E}_{\nu_h^N}\left[e^{|\int_I NG_N(t,\eta_{tN^\gamma})\,\mathrm{d}t|}\right]. \tag{3.23}$$

Since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_{N\to\infty}\tfrac{1}{N}\log(a_N+b_N) \leq \max\left\{\limsup_{N\to\infty}\tfrac{1}{N}\log(a_N),\limsup_{N\to\infty}\tfrac{1}{N}\log(b_N)\right\},$$

we can remove the absolute value from expression (3.23). By Feynman–Kac's formula (see Lemma 7.3 in [1]), we finally have that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y(tN^\gamma)-\eta_x(tN^\gamma))\mathrm{d}t\right]\\
&\leq C_0 + \int_I \sup_f\Bigg\{N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int(\eta_y-\eta_x)f(\eta)\mathrm{d}\nu_h^N\\
&\quad + N^{\gamma-1}\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_h^N}\Bigg\}\mathrm{d}t,
\end{aligned} \tag{3.24}$$

where the supremum is taken over all densities f on Ω_N with respect to ν_h^N. Note that, by a change of variables, we have that

$$\begin{aligned}
&N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t\left(\tfrac{x}{N},\tfrac{y}{N}\right)p(y-x)\int(\eta_y-\eta_x)f(\eta)\mathrm{d}\nu_h^N\\
&=N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int(\eta_y-\eta_x)f(\eta)\mathrm{d}\nu_h^N\\
&=N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int\eta_y\left(f(\eta)-f(\sigma^{x,y}\eta)\right)\mathrm{d}\nu_h^N\\
&\quad+N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int\eta_x f(\eta)\left(\theta^{x,y}(\eta)-1\right)\mathrm{d}\nu_h^N,
\end{aligned}\tag{3.25}$$

where $\theta^{x,y}(\eta)=\frac{\mathrm{d}\nu_h^N(\sigma^{x,y}\eta)}{\mathrm{d}\nu_h^N(\eta)}$ and F^a is the antisymmetric part of F, i.e. for all $t\in I$ and $(u,v)\in[0,1]^2$

$$F_t^a(u,v)=\frac{1}{2}\Big[F_t(u,v)-F_t(v,u)\Big].$$

Observe that $F_t^a(u,u)=0$. By Young's inequality, the fact that f is a density and $|\eta_y|\leq 1$, we have that, for any $A>0$, the third term in (3.25) is bounded from above by a constant times

$$\begin{aligned}
&N^{\gamma-1}A\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2p(y-x)+\frac{N^{\gamma-1}}{A}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}p(y-x)I_{x,y}(\sqrt{f},\nu_h^N)\\
&\leq\frac{c_\gamma A}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}}+\frac{2N^{\gamma-1}}{A}D_N^0(\sqrt{f},\nu_h^N).
\end{aligned}$$

Since h is Lipschitz we have that $\sup_{\eta\in\Omega_N}|\theta^{x,y}(\eta)-1|=\mathscr{O}\left(\frac{|x-y|}{N}\right)$. By Young's inequality and the fact that f is a density, for any $A'>0$, the last term in (3.25) is bounded from above by

$$\begin{aligned}
&\frac{N^{\gamma-1}}{A'}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2p(y-x)+A'N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}p(y-x)\left(\tfrac{|x-y|}{N}\right)^2\\
&=\frac{c_\gamma}{A'N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}}+\frac{A'c_\gamma}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{1}{|\tfrac{x}{N}-\tfrac{y}{N}|^{\gamma-1}}.
\end{aligned}$$

Recall (3.21), so that by choosing $A = 8$ and $B = 1$ and using the two results above we have just proved that (3.24) is bounded from above by C_0 plus

$$\frac{c_\gamma(8+\frac{1}{A'})}{N^2}\sum_{x\neq y\in\Lambda_N}\frac{\left[F_t^a(\frac{x}{N},\frac{y}{N})\right]^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}}+C(\kappa+1)+c_\gamma A^{'}A^{''},$$

where

$$A^{''}:=\sup_{\varepsilon>0}\sup_{N\geq 1}\frac{1}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{1}{|\frac{x}{N}-\frac{y}{N}|^{\gamma-1}}<\infty$$

since $\gamma < 2$. Therefore, we have proved that there exist constants $A^{'''}$ and $B^{'}$ (independent of $\varepsilon > 0$, $N \geq 1$, and $F \in C_c^\infty(I\times[0,1]^2)$) such that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}\Bigg[\int_I N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y^N(t)-\eta_x^N(t))\,\mathrm{d}t\Bigg]\\
&\quad=\mathbb{E}_{\mu_N}\left[\int_I -2c_\gamma\langle\pi_t^N,g_t^N\rangle\,\mathrm{d}t\right]\\
&\quad\leq\int_I\frac{A^{'''}}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{c_\gamma\left(F_t^a(\frac{x}{N},\frac{y}{N})\right)^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}}\mathrm{d}t+B^{'}|I|(\kappa+1).
\end{aligned}\tag{3.26}$$

Above the function g^N is defined on $I\times[0,1]$ by

$$g_t^N(u)=\frac{1}{N}\sum_{y\in\Lambda_N}\mathbf{1}_{\left|\frac{y}{N}-u\right|\geq\varepsilon}\frac{F_t^a\left(u,\frac{y}{N}\right)}{|u-\frac{y}{N}|^{1+\gamma}}$$

and it is a discretization of the smooth function g defined on $(t,u)\in I\times[0,1]$ by

$$g_t(u)=\int_0^1\mathbf{1}_{\{|v-u|\geq\varepsilon\}}\frac{F_t^a(u,v)}{|u-v|^{1+\gamma}}\mathrm{d}v.$$

Let $Q_\varepsilon=\{(u,v)\in[0,1]^2\ ;\ |u-v|\geq\varepsilon\}$. Observe first that for symmetry reasons we have that, for any integrable function π,

$$\int_0^1\pi(u)g_t(u)\mathrm{d}u=\iint_{Q_\varepsilon}\frac{(\pi(v)-\pi(u))F_t^a(u,v)}{|u-v|^{1+\gamma}}\mathrm{d}u\mathrm{d}v.$$

By taking the limit as $N\to\infty$ in (3.26), we conclude that there exist constants $C>0$ independent of $F\in C_c^{0,\infty}(I\times[0,1]^2)$ and $\varepsilon>0$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_I\iint_{Q_\varepsilon}\frac{(\pi_t^\kappa(v)-\pi_t^\kappa(u))F_t^a(u,v)}{|u-v|^{1+\gamma}}-C\frac{\left(F_t^a(u,v)\right)^2}{|u-v|^{1+\gamma}}\mathrm{d}u\mathrm{d}v\mathrm{d}t\right]\lesssim|I|(\kappa+1).$$

Using similar arguments as to the ones in the proof of Lemma 6.1 of [4], we can insert the supremum over F inside the expectation above, so that

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_F\left\{\int_I\iint_{Q_\varepsilon}\frac{(\pi_t^\kappa(v)-\pi_t^\kappa(u))F_t^a(u,v)}{|u-v|^{1+\gamma}}\right.\right.$$
$$\left.\left.-C\frac{\left(F_t^a(u,v)\right)^2}{|u-v|^{1+\gamma}}\,\mathrm{d}u\mathrm{d}v\mathrm{d}t\right\}\right]\lesssim|I|(\kappa+1).$$

Since the function $(u,v)\in[0,1]^2\to\pi(v)-\pi(u)$ is antisymmetric we may replace F^a by F in the previous variational formula, i.e.

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_F\left\{\int_I\iint_{Q_\varepsilon}\frac{(\pi_t^\kappa(v)-\pi_t^\kappa(u))F_t(u,v)}{|u-v|^{1+\gamma}}\right.\right.$$
$$\left.\left.-C\frac{\left(F_t(u,v)\right)^2}{|u-v|^{1+\gamma}}\,\mathrm{d}u\mathrm{d}v\mathrm{d}t\right\}\right]\lesssim|I|(\kappa+1). \tag{3.27}$$

Consider the Hilbert space $\mathbb{L}^2([0,1]^2,\mathrm{d}\mu_\varepsilon)$ where μ_ε is the measure whose density with respect to Lebesgue measure is given by $(u,v)\in[0,1]^2\to\mathbb{1}_{|u-v|\geq\varepsilon}\,|u-v|^{-(1+\gamma)}$. By taking

$$\Pi^\kappa:(t;u,v)\in I\times[0,1]^2\to\pi_t^\kappa(v)-\pi_t^\kappa(u),$$

the previous formula implies that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_I\iint_{[0,1]^2}\left(\Pi_t^\kappa(u,v)\right)^2\,\mathrm{d}\mu_\varepsilon(u,v)\mathrm{d}t\right]\lesssim|I|(\kappa+1). \tag{3.28}$$

Letting $\varepsilon\to0$, by the monotone convergence theorem, we conclude that

$$\int_I\iint_{[0,1]^2}\frac{(\pi_t^\kappa(v)-\pi_t^\kappa(u))^2}{|u-v|^{1+\gamma}}\,\mathrm{d}u\mathrm{d}v\mathrm{d}t<\infty,$$

$\mathbb{Q}$ almost surely.

Second item: $\displaystyle\int_I\int_0^1\left\{\frac{(\alpha-\pi_t^\kappa(u))^2}{u^\gamma}+\frac{(\beta-\pi_t^\kappa(u))^2}{(1-u)^\gamma}\right\}\mathrm{d}u\;\mathrm{d}t<\infty$ $\mathbb{Q}$ almost surely. Now we have to prove that the function $(t,u)\to\pi_t^\kappa(u)-\alpha$ is in the space $L^2(I\times(0,1),\mathrm{d}t\otimes\mathrm{d}\mu)$, where μ is the measure whose density with respect to the Lebesgue measure is given by

$$u\in(0,1)\to\frac{1}{u^\gamma}.$$

A similar argument to the one used in the first item shows that the function $(t,u)\to\pi_t^\kappa(u)-\beta$ belongs to $L^2([0,T]\times(0,1),\mathrm{d}t\otimes\mathrm{d}\mu')$, where μ' is the measure whose density with respect to the Lebesgue measure is given by

$$u\in[0,1]\to\frac{1}{(1-u)^\gamma}.$$

Let ν_h^N be the Bernoulli product measure corresponding to a profile h which is Lipschitz such that $h(0) = \alpha \leq h(u) \leq \beta = h(1)$ for all $u \in [0, 1]$. Let $G \in C_c^\infty(I \times [0, 1])$. As in the beginning of the proof of Theorem 3.2, using the entropy and Jensen's inequalities we get that

$$\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1} \sum_{x\in\Lambda_N} G_t r_N^-\left(\tfrac{x}{N}\right)(\eta_x(t\Theta(N)) - \alpha)\mathrm{d}t\right] \\ \leq \frac{H_N(\mu_N|\nu_h^N)}{N} + \frac{1}{N}\log \mathbb{E}_{\mu_N}\left[e^{N|\int_I N^{\gamma-1}\sum_{x\in\Lambda_N} G_t r_N^-\left(\frac{x}{N}\right)(\eta_x(t\Theta(N))-\alpha)\mathrm{d}t|}\right].$$

Now, using (3.17) and Feynman–Kac's formula (see Lemma 7.3 of [1]) we obtain that

$$\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1} \sum_{x\in\Lambda_N} G_t r_N^-\left(\tfrac{x}{N}\right)(\eta_x(t\Theta(N)) - \alpha)\mathrm{d}t\right] \\ \leq C_0 + \int_I \sup_f \left\{ N^{\gamma-1} \sum_{x\in\Lambda_N} (G_t r_N^-)\left(\tfrac{x}{N}\right)\langle \eta_x - \alpha, f\rangle_{\nu_h^N} \right. \\ \left. +\Theta(N)N^{-1}\left\langle L_N\sqrt{f}, \sqrt{f}\right\rangle_{\nu_h^N}\right\}\mathrm{d}t, \tag{3.29}$$

where the supremun is taken over all the densities f on Ω_N with respect to ν_h^N. Using (3.21) with $B = 1$ we can bound from above the second term on the right hand side of (3.29) by

$$-\frac{\Theta(N)}{4N} D_N(\sqrt{f}, \nu_h^N) + C\Theta(N)N^{-\gamma}(\kappa N^{-\theta} + 1),$$

and from Lemma 3.5 with $A_x = \frac{1}{\kappa} G_t\left(\frac{x}{N}\right)$ the first term on the right side of (3.29) is bounded from above by

$$\frac{CN^{\gamma-1}}{\kappa} \sum_{x\in\Lambda_N} r_N^-\left(\tfrac{x}{N}\right)\left(G_t\left(\tfrac{x}{N}\right)\right)^2 + C(\kappa+1).$$

Taking $N \to \infty$ we can conclude that there exists a constant $C' > 0$ independent of G and of t such that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_I\int_0^1 \left(\frac{(\pi_t^\kappa(u) - \alpha)G_t(u)}{|u|^\gamma} - \frac{C'}{\kappa}\frac{G_t^2(u)}{|u|^\gamma}\right)\mathrm{d}u\mathrm{d}t\right] \lesssim |I|(\kappa+1).$$

From Lemma 6.1 in [4] we can insert the supremum over G inside the expectation above, and we get

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{G}\left\{\int_I\int_0^1\left(\frac{(\pi_t^\kappa(u)-\alpha)G_t(u)}{|u|^\gamma}-\frac{C'}{\kappa}\frac{G_t^2(u)}{|u|^\gamma}\right)\mathrm{d}u\mathrm{d}t\right\}\right]\lesssim|I|(\kappa+1). \tag{3.30}$$

The previous formula implies that

$$\int_I\int_0^1\frac{(\pi_t^\kappa(u)-\alpha)^2}{|u|^\gamma}\,\mathrm{d}u\mathrm{d}t<\infty,$$

$\mathbb{Q}$ almost surely. Similarly, we get

$$\int_I\int_0^1\frac{(\pi_t^\kappa(u)-\beta)^2}{|u|^\gamma}\,\mathrm{d}u\mathrm{d}t<\infty,$$

$\mathbb{Q}$ almost surely.

Conclusion. By Definition 2.3, the two steps above allow us to show that $\mathbb{Q}$ is concentrated on trajectories of measures whose density is a weak solution of the corresponding hydrodynamic equation (see Proposition 3.6). By uniqueness of the weak solution (see Lemma 2.8) we get that $\mathbb{Q}$ is unique. Indeed, we have that $\mathbb{Q}=\delta_{\{\rho_t^\kappa(u)du\}}$ (Dirac mass). Then, by using the latter, we compute the expectation in (3.28) and (3.30) and we are done. □

3.4. Characterization of Limit Points

In the present subsection we characterize all limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\geq1}$, which we know that exist from the results of Section 3.2. Let us assume without loss of generality, that $\{\mathbb{Q}_N\}_{N\geq1}$ converges to $\mathbb{Q}$. Since there is at most one particle per site, it is easy to show that $\mathbb{Q}$ is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure, i.e. $\pi_t^\kappa(du)=\rho_t^\kappa(u)\mathrm{d}u$. Indeed, for any $t\in[0,T]$ and for any function $G:[0,1]\to\mathbb{R}$ we have that

$$|\langle\pi_t^N,G\rangle|\leq\frac{1}{N-1}\sum_{x\in\Lambda_N}|G(\tfrac{x}{n})|.$$

Also, we know that for any continuous function G the functional $\pi\in\mathscr{M}^+\to\langle\pi,G\rangle$ is continuous. Then, taking $N\to\infty$, we obtain that

$$|\langle\pi_t,G\rangle|\leq\int_0^1|G(u)|\mathrm{d}u.$$

which implies that π_t is absolutely continuous with respect to the Lebesgue measure. In Proposition 3.6 below we prove, for each range of θ, that $\mathbb{Q}$ is concentrated on trajectories of measures whose density satisfies a weak form of the corresponding hydrodynamic equation. Moreover, we have seen in Theorem 3.2 that $\mathbb{Q}$ is concentrated on trajectories of measures whose density satisfies the energy estimate, i.e. $\rho^\kappa\in L^2(0,T;\mathscr{H}^{\gamma/2})$ when $\theta=0$ and

$$\int_0^T\int_0^1\left\{\frac{(\alpha-\rho_t^\kappa(u))^2}{u^\gamma}+\frac{(\beta-\rho_t^\kappa(u))^2}{(1-u)^\gamma}\right\}\mathrm{d}u\mathrm{d}t<\infty$$

for any $\theta \leq 0$. Since a weak solution of the hydrodynamic equation (2.10) is unique we have that $\mathbb{Q}$ is unique and takes the form of a Dirac mass.

Proposition 3.6. *If $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$ then*

1. *if $\theta < 0$:*

$$\mathbb{Q}\Big(\pi_\cdot : F_{Reac}(t, \rho^\kappa, G, g) = 0, \forall t \in [0,T],\ \forall G \in C_c^{1,2}([0,T]\times[0,1])\Big) = 1.$$

2. *if $\theta = 0$:*

$$\mathbb{Q}\Big(\pi_\cdot : F_{Dir}(t, \rho^\kappa, G, g) = 0, \forall t \in [0,T],\ \forall G \in C_c^{1,2}([0,T]\times[0,1])\Big) = 1.$$

Proof. Note that in order to prove the proposition, it is enough to verify, for $\delta > 0$ and G in the corresponding space of test functions, that

$$\mathbb{Q}\left(\pi_\cdot \in \mathscr{D}^T_{\mathscr{M}^+} : \sup_{0\leq t\leq T} \big|F_\theta(t, \rho^\kappa, G, g)\big| > \delta\right) = 0,$$

for each θ, where F_θ stands for F_{Reac} if $\theta < 0$ and F_{Dir} if $\theta = 0$. Indeed, we have that

$$\begin{aligned} F_\theta(t, \rho^\kappa, G, g) &= \big\langle \rho^\kappa_t, G_t\big\rangle - \langle g, G_0\rangle - \int_0^t \Big\langle \rho^\kappa_s, \Big(\partial_s + \mathbb{1}_{\{\theta=0\}}\mathbb{L}\Big) G_s\Big\rangle \mathrm{d}s \\ &\quad + \mathbb{1}_{\{\theta\leq 0\}}\kappa \int_0^t \big\langle \rho^\kappa_s, G_s\big\rangle_{V_1} \mathrm{d}s - \mathbb{1}_{\{\theta\leq 0\}}\kappa \int_0^t \langle G_s, V_0\rangle\ \mathrm{d}s = 0. \end{aligned} \tag{3.31}$$

From here on, in order to simplify notation, we will erase $\pi_\cdot$ from the sets that we have to look at.

By definition of F_θ above we can bound from above the previous probability by the sum of

$$\mathbb{Q}\left(\sup_{0\leq t\leq T} \big|F_\theta(t, \rho^\kappa, G, \rho_0)\big| > \frac{\delta}{2}\right) \tag{3.32}$$

and

$$\mathbb{Q}\left(|\langle \rho_0 - g, G_0\rangle| > \frac{\delta}{2}\right).$$

We note that last probability is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$ and $\mathbb{Q}_N$ is induced by μ_N which is associated to g. Now we deal with (3.32). Since for $\theta \leq 0$ the function G_s has compact support included in $(0, 1)$ the singularities of V_0 and V_1 are not present, thus from Proposition A.3 of [14], the set inside the probability in (3.32) is an open set in the Skorohod topology. Therefore, from Portmanteau's Theorem we bound (3.32) from above by

$$\liminf_{N\to\infty} \mathbb{Q}_N\left(\sup_{0\leq t\leq T} \big|F_\theta(t, \rho^\kappa, G, \rho_0)\big| > \frac{\delta}{2}\right).$$

Summing and subtracting $\int_0^t \Theta(N) L_N \langle \pi_s^N, G_s \rangle \mathrm{d}s$ to the term inside the previous absolute value, recalling (3.1) and the definition of $\mathbb{Q}_N$, we can bound the previous probability from above by the sum of the next two terms

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|M_t^N(G)\right| > \frac{\delta}{4}\right)$$

and

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|\int_0^t \Theta(N) L_N \langle \pi_s^N, G_s\rangle \mathrm{d}s - \int_0^t \left\langle \pi_s^N, \mathbb{1}_{\{\theta=0\}} \mathbb{L} G_s \right\rangle \mathrm{d}s \right.\right.$$
$$\left.\left. + \mathbb{1}_{\{\theta\leq 0\}}\kappa \int_0^t \langle \rho_s, G_s\rangle_{V_1} \,\mathrm{d}s \; - \mathbb{1}_{\{\theta\leq 0\}}\kappa \int_0^t \langle G_s, V_0\rangle \,\mathrm{d}s\right| > \frac{\delta}{4}\right). \tag{3.33}$$

By Doob's inequality we have that

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|M_t^N(G)\right| > \frac{\delta}{4}\right) \lesssim \frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[\int_0^T \Theta(N)\left[L_N \langle \pi_s^N, G\rangle^2\right.\right.$$
$$\left.\left. - 2\langle \pi_s^N, G\rangle L_N \langle \pi_s^N, G\rangle\right] \mathrm{d}s\right].$$

In the proof of Proposition 3.1 we have proved that the term inside the time integral in the previous expression is $\mathscr{O}(N^{\gamma-2})$. Then, using the fact that $\gamma < 2$ we have that last probability vanishes as $N \to \infty$. It remains to prove that (3.33) vanishes as $N \to \infty$. For that purpose, we recall (3.2) and we bound (3.33) from above by the sum of the following terms:

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|\int_0^t \frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N} \mathscr{L}_N G_s(\tfrac{x}{N})\eta_x^N(s)\mathrm{d}s\right.\right.$$
$$\left.\left. - \int_0^t \left\langle \pi_s^N, \mathbb{1}_{\{\theta=0\}}\mathbb{L} G_s\right\rangle \mathrm{d}s\right| > \frac{\delta}{2^4}\right), \tag{3.34}$$

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|\int_0^t \left\{\frac{\kappa\Theta(N)}{N^\theta(N-1)}\sum_{x\in\Lambda_N}(G_s r_N^-)(\tfrac{x}{N})(\alpha - \eta_x^N(s))\right.\right.\right.$$
$$\left.\left.\left. - \mathbb{1}_{\{\theta\leq 0\}}\kappa\int_0^1 (G_s r^-)(u)(\alpha - \rho_s^\kappa(u))\mathrm{d}u\right\}\mathrm{d}s\right| > \frac{\delta}{2^4}\right) \tag{3.35}$$

and

$$\mathbb{P}_{\mu_N}\left(\sup_{0\leq t\leq T}\left|\int_0^t \left\{\frac{\kappa\Theta(N)}{N^\theta(N-1)}\sum_{x\in\Lambda_N}(G_s r_N^+)(\tfrac{x}{N})(\beta - \eta_x^N(s))\right.\right.\right. \tag{3.36}$$
$$\left.\left.\left. - \mathbb{1}_{\{\theta\leq 0\}}\kappa\int_0^1 (G_s r^+)(u)(\beta - \rho_s^\kappa(u))\mathrm{d}u\right\}\mathrm{d}s\right| > \frac{\delta}{2^4}\right). \tag{3.37}$$

For $\theta = 0$ from (3.6) we have that (3.34) goes to 0 as $N \to \infty$. For $\theta \le 0$ we have that from (3.6) and 3.5 the boundary terms (3.35) and (3.36) go to 0 as $N \to \infty$. This finishes the proof Proposition 3.6. □

4. Proof of Theorem 2.13

For easy understanding of the proof of items (i) and (ii) of Theorem 2.13, we first establish some notation and prove some lemmata.

Recall the function $\bar\rho^\infty$ introduced in Remark 2.7 which can be rewritten as

$$\bar\rho^\infty(u) = \frac{\beta u^\gamma + \alpha(1-u)^\gamma}{u^\gamma + (1-u)^\gamma}.$$

It is easy to see that $\bar\rho^\infty(0) = \alpha$ and $\bar\rho^\infty(1) = \beta$. Moreover, it is not difficult to see that $\bar\rho^\infty \in C^1([0,1])$ and that

$$\lim_{u\to 0}(\bar\rho^\infty(u))' u^{2-\gamma} = \lim_{u\to 1}(\bar\rho^\infty(u))'(1-u)^{2-\gamma} = 0,$$

and from Lemma 7.2 of [15] we conclude that

$$\|\bar\rho^\infty\|_{\gamma/2} < \infty. \tag{4.1}$$

By the fractional Hardy's inequality (see e.g. [12]) and the fact that $V_1(\frac{1}{2}) \le V_1(u)$ for all $u \in (0,1)$ we know that

$$\|g\| \lesssim \|g\|_{V_1} \lesssim \|g\|_{\gamma/2} \tag{4.2}$$

for any $g \in \mathcal{H}_0^{\gamma/2}$, and where $\|g\|_{V_1}$ is defined in the beginning of Section 2.3.

In order to prove items (i) and (ii) of Theorem 2.13 we first guarantee the existence of weak solutions of Equation (2.10) with $\kappa = 0$ and (2.12), (see Lemmas 4.1 and 4.3 below), then we establish the convergence in $L^2(0,T;L^2)$ (see Lemmas 4.2 and 4.4) which will allow us to conclude.

Lemma 4.1. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Then, there exists a weak solution of (2.10) with $\hat\kappa = 0$ and initial condition ρ_0.*

Proof. The strategy of the proof is to construct the solution as the limit of ρ^κ, as $\kappa \to 0$, where ρ^k is the weak solution of (2.10) with initial condition ρ_0 and $\hat\kappa = \kappa$.

By item i) in Theorem 3.2, for any $\kappa > 0$ we know that

$$\int_I \|\rho_t^\kappa\|_{\gamma/2}^2 \mathrm{d}t \lesssim |I|(\kappa+1) \tag{4.3}$$

for any interval $I \subset [0,T]$. We define

$$\forall t \in [0,T], \quad \forall u \in [0,1], \quad \varphi_t^\kappa(u) := \rho_t^\kappa(u) - \bar\rho^\infty(u). \tag{4.4}$$

Since we are interested in small values of κ, say $\kappa \leq 1$, from (4.3), (4.1) and the fact $(a+b)^2 \leq 2a^2 + 2b^2$, it is not difficult to see that

$$\int_I \|\varphi_t^\kappa\|^2_{\gamma/2} \mathrm{d}t \lesssim |I|, \tag{4.5}$$

thus we have that $\varphi^\kappa \in L^2(0, T; \mathscr{H}_0^{\gamma/2})$. It is also easy to see that φ^κ satisfies

$$\begin{aligned} \langle \varphi_t^\kappa, G_t \rangle - \langle \varphi_0, G_0 \rangle - \int_0^t \left\langle \varphi_s^\kappa, (\mathbb{L} + \partial_s)\, G_s \right\rangle \mathrm{d}s \\ + \kappa \int_0^t \langle \varphi_s^\kappa, G_s \rangle_{V_1} \mathrm{d}s - \int_0^t \langle \bar{\rho}^\infty, \mathbb{L} G_s \rangle \mathrm{d}s = 0 \end{aligned} \tag{4.6}$$

for all $t \in [0, T]$, for any function $G \in C_c^{1,\infty}([0, T] \times (0, 1))$ and where $\varphi_0(u) = \rho_0(u) - \bar{\rho}^\infty(u)$. From (4.5) we conclude that there exists a subsequence of $(\varphi^\kappa)_{\kappa \in (0,1)}$ converging weakly to some element $\varphi^0 \in L^2(0, T; \mathscr{H}_0^{\gamma/2})$ as $\kappa \to 0$. We claim that $\rho^0 := \bar{\rho}^\infty + \varphi^0$ is the desired solution. Indeed, first note that since the norm $\| \cdot \|_{\gamma/2}$ is weakly lower-semicontinuous we have that

$$\int_I \|\varphi_t^0\|^2_{\gamma/2} \mathrm{d}t \lesssim |I|. \tag{4.7}$$

By using $(a+b)^2 \leq 2a^2 + 2b^2$ we have that

$$\int_I \|\rho_t^0\|^2_{\gamma/2} \mathrm{d}t \leq 2 \int_I \|\bar{\rho}^\infty\|^2_{\gamma/2} \mathrm{d}t + 2 \int_I \|\varphi_t^0\|^2_{\gamma/2} \mathrm{d}t \lesssim |I|.$$

Taking $I = [0, T]$, we have that ρ^0 satisfies item i) of Definition 2.3. Since $\varphi^0 \in L^2(0, T; \mathscr{H}_0^{\gamma/2})$, it is easy to see that $\rho_t^0(0) = \bar{\rho}^\infty(0) = \alpha$ and $\rho_t^0(1) = \bar{\rho}^\infty(1) = \beta$ for almost every $t \in [0, T]$. Then, item (ii) for $\hat{\kappa} = 0$ in Definition 2.3 is satisfied. In order to verify that ρ^0 satisfies item (iii) in Definition 2.3 we first integrate (4.6) over $[0, t]$. Thus we have that

$$\begin{aligned} \int_0^t \langle \varphi_s^\kappa, G_s \rangle \mathrm{d}s - t \langle \varphi_0, G_0 \rangle - \int_0^t \int_0^s \left\langle \varphi_r^\kappa, (\mathbb{L} + \partial_r)\, G_r \right\rangle \mathrm{d}r \mathrm{d}s \\ + \kappa \int_0^t \int_0^s \langle \varphi_r^\kappa, G_r \rangle_{V_1} \mathrm{d}r \mathrm{d}s - \int_0^t \int_0^s \langle \bar{\rho}^\infty, \mathbb{L} G_r \rangle \mathrm{d}r \mathrm{d}s = 0 \end{aligned}$$

for any function $G \in C_c^{1,\infty}([0, T] \times (0, 1))$. Taking $\kappa \to 0$, by weak convergence and Lebesgue's dominated convergence theorem we get from the previous equality that

$$\int_0^t \langle \varphi_s^0, G_s \rangle \mathrm{d}s - t \langle \varphi_0, G_0 \rangle - \int_0^t \int_0^s \left\langle \varphi_r^0, (\mathbb{L} + \partial_r)\, G_r \right\rangle - \langle \bar{\rho}^\infty, \mathbb{L} G_r \rangle \mathrm{d}r \mathrm{d}s = 0.$$

Now, taking the derivative with respect to t in the previous equality we get that φ^0 satisfies

$$\langle \varphi_t^0, G_t \rangle - \langle \varphi_0, G_0 \rangle - \int_0^t \langle \varphi_s^0, \left(\mathbb{L} + \partial_s\right) G_s \rangle \, \mathrm{d}s - \int_0^t \langle \bar{\rho}^\infty, \mathbb{L} G_s \rangle \mathrm{d}s = 0 \tag{4.8}$$

for all $t \in [0, T]$. Then, item (iii) with $\kappa = 0$ in Definition 2.3 follows from (4.8), the definition of ρ^0 and $\bar{\rho}^\infty$ □

Lemma 4.2. *Let* $\rho_0 : [0,1] \to [0,1]$ *be a measurable function. Let* ρ^κ *be the weak solution of* (2.10) *with initial condition* ρ_0 *and* $\hat\kappa = \kappa$. *Then,* ρ^κ *converges strongly to* ρ^0 *in* $L^2(0,T;L^2)$ *as* κ *goes to* 0, *where* ρ^0 *is the weak solution of* (2.10) *with* $\hat\kappa = 0$ *and initial condition* ρ_0.

Proof. Note that is enough to show that

$$\int_0^t \|\rho_s^\kappa - \rho_s^0\|^2 \,\mathrm{d}s \lesssim t^2\kappa$$

for all $t \in [0,T]$. By Lemma 4.1 we know that $\rho^0 = \bar\rho^\infty + \varphi^0$. Then, last inequality is equivalent to

$$\int_0^t \|\varphi_s^\kappa - \varphi_s^0\|^2 \,\mathrm{d}s \lesssim t^2\kappa. \tag{4.9}$$

By subtracting (4.8) from (4.6) and calling $\delta_t^k := \varphi_t^\kappa - \varphi_t^0$ we obtain that

$$\langle \delta_t^\kappa, G_t\rangle - \int_0^t \left\langle \delta_s^\kappa, (\mathbb{L} + \partial_s)\, G_s\right\rangle \mathrm{d}s = -\kappa \int_0^t \langle \varphi_s^\kappa, G_s\rangle_{V_1} \mathrm{d}s \tag{4.10}$$

for any function $G \in C_c^{1,\infty}([0,T]\times(0,1))$. Let $\{H_n^\kappa\}_{n\geq 1}$ be a sequence of functions in $C_c^{1,\infty}([0,T]\times(0,1))$ converging to δ^κ as $n \to \infty$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$ and for $n \geq 1$, let $G_n^\kappa(s,u) = \int_s^t H_n^\kappa(r,u)\mathrm{d}r$. We claim that by plugging G_n into (4.10) and taking $n \to \infty$ we get that

$$\int_0^t \|\delta_s^\kappa\|^2 \,\mathrm{d}s + \frac{1}{2}\left\| \int_0^t \delta_s^\kappa \mathrm{d}s \right\|_{\gamma/2}^2 = -\kappa \int_0^t \left\langle \varphi_s^\kappa, \int_s^t \delta_r^\kappa \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s. \tag{4.11}$$

We leave the justification of the equality above to the end of the proof. Now, by using successively the Cauchy–Schwarz's inequality we have that

$$\begin{aligned}\int_0^t \|\delta_s^\kappa\|^2 \,\mathrm{d}s + \frac{1}{2}\left\| \int_0^t \delta_s^\kappa \mathrm{d}s \right\|_{\gamma/2}^2 &\leq \kappa \int_0^t \|\varphi_s^\kappa\|_{V_1} \left\| \int_s^t \delta_r^\kappa \mathrm{d}r \right\|_{V_1} \mathrm{d}s \\ &\lesssim \kappa \sqrt{\int_0^t \|\varphi_s^\kappa\|_{\gamma/2}^2 \mathrm{d}s} \sqrt{\int_0^t \left\| \int_s^t \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 \mathrm{d}s}.\end{aligned} \tag{4.12}$$

In the last inequality of the previous expression we used (4.2). By the triangular inequality we have that $\sqrt{\int_0^t \left\| \int_s^t \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 \mathrm{d}s}$ is bounded from above by

$$\begin{aligned}\sqrt{\int_0^t \left(\int_s^t \|\delta_r^\kappa\|_{\gamma/2} \mathrm{d}r \right)^2 \mathrm{d}s} &\leq \sqrt{t \int_0^t \int_0^t \|\delta_r^\kappa\|_{\gamma/2}^2 \mathrm{d}r\mathrm{d}s} \\ &\lesssim \sqrt{t^2 \int_0^t \left(\|\varphi_r^\kappa\|_{\gamma/2}^2 + \|\varphi_r^0\|_{\gamma/2}^2 \right) \mathrm{d}r}.\end{aligned} \tag{4.13}$$

In the first inequality in the previous display we used the Cauchy–Schwarz's inequality and in the second inequality we used the Minkowski's inequality and the inequality $(a+b)^2 \leq 2(a^2+b^2)$. Using (4.5) and (4.7), we get from (4.12) and (4.13) the result.

We conclude this proof justifying (4.11). Note that it is enough to show

(i) $\lim_{n\to\infty} \int_0^t \langle \delta_s^\kappa, (\partial_s G_n^\kappa)(s,\cdot)\rangle \mathrm{d}s = -\int_0^t \|\delta_s^\kappa\|^2 \mathrm{d}s.$

(ii) $\lim_{n\to\infty} \int_0^t \langle \delta_s^\kappa, \mathbb{L} G_n^\kappa(s,\cdot)\rangle \mathrm{d}s = -\frac{1}{2}\Big\| \int_0^t \delta_s^\kappa \mathrm{d}s \Big\|_{\gamma/2}^2.$

(iii) $\lim_{n\to\infty} \int_0^t \big\langle \varphi_s^\kappa, G_n^\kappa(s,\cdot)\big\rangle_{V_1} \mathrm{d}s = \int_0^t \Big\langle \varphi_s^\kappa, \int_s^t \delta_r^\kappa \mathrm{d}r \Big\rangle_{V_1} \mathrm{d}s.$

For (i) we rewrite $\int_0^t \langle \delta_s^\kappa, (\partial_s G_n^\kappa)(s,\cdot)\rangle \mathrm{d}s$ as

$$-\int_0^t \langle \delta_s^\kappa\,,\, H_n^\kappa(s,\cdot)\rangle \,\mathrm{d}s = -\int_0^t \big\langle \delta_s^\kappa\,,\, H_n^\kappa(s,\cdot)-\delta_s^\kappa\big\rangle \,\mathrm{d}s - \int_0^t \|\delta_s^\kappa\|^2 \,\mathrm{d}s.$$

Observe then that by Cauchy–Schwarz's inequality we have

$$\begin{aligned}\left|\int_0^T \big\langle \delta_s^\kappa\,,\, H_n^\kappa(s,\cdot)-\delta_s^\kappa\big\rangle \mathrm{d}s\right| &\leq \int_0^T \|\delta_s^\kappa\|\,\|H_n^\kappa(s,\cdot)-\delta_s^\kappa\|\,\mathrm{d}s \\ &\leq \sqrt{\int_0^T \|\delta_s^\kappa\|^2\,\mathrm{d}s}\,\sqrt{\int_0^T \|H_n^\kappa(s,\cdot)-\delta_s^\kappa\|^2\,\mathrm{d}s},\end{aligned}$$

which goes to 0 as $n\to\infty$ since $H_n^\kappa \to \delta_s^\kappa$ in $L^2(0,T;\mathscr{H}_0^{\gamma/2})$. For (ii), since G_n has compact support included in $(0,1)$, we can use the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [15]) which permits us to write

$$\int_0^t \langle \delta_s^\kappa, \mathbb{L} G_n^\kappa(s,\cdot)\rangle \mathrm{d}s = -\int_0^t \Big\langle \delta_s^\kappa, G_n^\kappa(s,\cdot)\Big\rangle_{\gamma/2} \mathrm{d}s.$$

Then we have

$$\begin{aligned}\int_0^t \Big\langle \delta_s^\kappa\,, G_n^\kappa(s,\cdot)\Big\rangle_{\gamma/2} \mathrm{d}s &= \int_0^t \Big\langle \delta_s^\kappa\,, \int_s^t \delta_r^\kappa \mathrm{d}r\Big\rangle_{\gamma/2} \mathrm{d}s \\ &\quad + \int_0^t \Big\langle \delta_s^\kappa\,, G_n^\kappa(s,\cdot) - \int_s^t \delta_r^\kappa \mathrm{d}r\Big\rangle_{\gamma/2} \mathrm{d}s \\ &= \iint_{0\leq s<r\leq t} \langle \delta_s^\kappa\,,\, \delta_r^\kappa\rangle_{\gamma/2}\,\mathrm{d}s\mathrm{d}r + \int_0^t \Big\langle \delta_s^\kappa\,, \int_s^t \big(H_n^\kappa(r,\cdot)-\delta_r^\kappa\big)\,\mathrm{d}r\Big\rangle_{\gamma/2} \mathrm{d}s \\ &= \frac{1}{2}\iint_{[0,t]^2} \langle \delta_s^\kappa\,,\, \delta_r^\kappa\rangle_{\gamma/2}\,\mathrm{d}s\mathrm{d}r + \int_0^t \Big\langle \delta_s^\kappa\,, \int_s^t \big(H_n^\kappa(r,\cdot)-\delta_r^\kappa\big)\,\mathrm{d}r\Big\rangle_{\gamma/2} \mathrm{d}s \\ &= \frac{1}{2}\Big\|\int_0^t \delta_s^\kappa \mathrm{d}s\Big\|_{\gamma/2}^2 + \int_0^t \Big\langle \delta_s^\kappa\,, \int_s^t \big(H_n^\kappa(r,\cdot)-\delta_r^\kappa\big)\,\mathrm{d}r\Big\rangle_{\gamma/2} \mathrm{d}s.\end{aligned}$$

To conclude the proof of (ii) it is sufficient to show that the term at the right hand side of last expression vanishes as n goes to ∞. This is a consequence of a successive use of Cauchy–Schwarz's inequalities:

$$\begin{aligned}
&\left|\int_0^t \left\langle \delta_s^\kappa , \int_s^t \left(H_n^\kappa(r,\cdot) - \delta_r^\kappa\right) \mathrm{d}r \right\rangle_{\gamma/2} \mathrm{d}s\right| \\
&\quad \le \int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \left\| \int_s^t \left(H_n^\kappa(r,\cdot) - \delta_r^\kappa\right) \mathrm{d}r \right\|_{\gamma/2} \mathrm{d}s \\
&\quad \le \int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \int_s^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{\gamma/2} \mathrm{d}r\, \mathrm{d}s \\
&\quad \le \int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{\gamma/2} \mathrm{d}r\, \mathrm{d}s \\
&\quad = \left(\int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \mathrm{d}s\right) \left(\int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{\gamma/2} \mathrm{d}r\right) \\
&\quad \le t \sqrt{\int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2}^2 \mathrm{d}s} \sqrt{\int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{\gamma/2}^2 \mathrm{d}r} \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{4.14}$$

To prove iii) we rewrite $\int_0^t \langle \varphi_s^\kappa, G_n^\kappa(s,\cdot)\rangle_{V_1} \mathrm{d}s$ as

$$\int_0^t \left\langle \varphi_s^\kappa , \int_s^t \left(H_n^\kappa(r,\cdot) - \delta_r^\kappa\right) \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s + \int_0^t \left\langle \varphi_s^\kappa , \int_s^t \delta_r^\kappa \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s$$

and, to conclude the proof, it is sufficient to show that the term at the left hand side of last expression vanishes as $n \to \infty$. This is a consequence of a successive use of the Cauchy–Schwarz's inequality as in (4.14), with $\|\cdot\|_{\gamma/2}$ replaced by $\|\cdot\|_{V_1}$ and Hardy's inequality:

$$\begin{aligned}
&\left|\int_0^t \left\langle \varphi_s^\kappa , \int_s^t \{H_n^\kappa(r,\cdot) - \delta_r^\kappa\} \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s\right| \\
&\quad \le \int_0^t \left\|\varphi_s^\kappa\right\|_{V_1} \left\| \int_s^t \left(H_n^\kappa(r\cdot) - \delta_r^\kappa\right) \mathrm{d}r \right\|_{V_1} \mathrm{d}s \\
&\quad \le \int_0^t \left\|\varphi_s^\kappa\right\|_{V_1} \int_s^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{V_1} \mathrm{d}r\, \mathrm{d}s \\
&\quad \le \int_0^t \left\|\varphi_s^\kappa\right\|_{V_1} \int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{V_1} \mathrm{d}r\, \mathrm{d}s \\
&\quad = \left(\int_0^t \left\|\varphi_s^\kappa\right\|_{V_1} \mathrm{d}s\right) \left(\int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{V_1} \mathrm{d}r\right) \\
&\quad \le t \sqrt{\int_0^t \left\|\varphi_s^\kappa\right\|_{V_1}^2 \mathrm{d}s} \sqrt{\int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{V_1}^2 \mathrm{d}r} \\
&\quad \le Ct \sqrt{\int_0^t \left\|\varphi_s^\kappa\right\|_{\gamma/2}^2 \mathrm{d}s} \sqrt{\int_0^t \left\| H_n^\kappa(r,\cdot) - \delta_r^\kappa \right\|_{\gamma/2}^2 \mathrm{d}r} \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

where in the last inequality we used the fractional Hardy's inequality (see (4.2)). □

Lemma 4.3. *Let* $\rho_0 : [0,1] \to [0,1]$ *be a measurable function. Consider the function* $\rho_t^\infty = \bar{\rho}^\infty + (\rho_0 - \bar{\rho}^\infty)e^{-tV_1}$. *If* $g^\infty := \rho_0 - \bar{\rho}^\infty \in \mathscr{H}^{\gamma/2}$, *then*

(i) $\rho^\infty \in L^2(0,T;\mathscr{H}^{\gamma/2})$;
(ii) ρ^∞ *is a weak solution of* (2.12) *with initial condition* ρ_0.

Proof. For (i) note that by using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ we get that

$$\int_0^T \|\rho_t^\infty\|_{\gamma/2}^2 \mathrm{d}t \le 2T\|\bar{\rho}^\infty\|_{\gamma/2}^2 + 2\int_0^T \left\| g^\infty e^{-tV_1}\right\|_{\gamma/2}^2 \mathrm{d}t.$$

Since $\|\bar{\rho}^\infty\|_{\gamma/2} < \infty$ (see (4.1)) it is enough to prove that the term on the right hand side of last expression is finite. Note that $\left\|g^\infty e^{-tV_1}\right\|_{\gamma/2}^2$ is equal to

$$\frac{c_\gamma}{2}\iint_{[0,1]^2} \frac{\left(g^\infty(u)e^{-tV_1(u)} - g^\infty(v)e^{-tV_1(v)}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v$$
$$= \frac{c_\gamma}{2}\iint_{[0,1]^2} \frac{\left(g^\infty(u)\left(e^{-tV_1(u)} - e^{-tV_1(v)}\right) + \left(g^\infty(u) - g^\infty(v)\right)e^{-tV_1(v)}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v.$$

Using the fact that $(a+b)^2 \le 2a^2 + 2b^2$ and that $|g^\infty(u)| \le 2$ for any $u \in [0,1]$ we get that last expression is less than $8\|e^{-tV_1}\|_{\gamma/2}^2 + 2\|g^\infty\|_{\gamma/2}^2$. Note that the term $8\|e^{-tV_1}\|_{\gamma/2}^2$ can be written as

$$4c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u -tV_1'(w)e^{-tV_1(w)}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v$$
$$= 4c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u t\left(\frac{\gamma}{w}r^-(w) - \frac{\gamma}{1-w}r^+(w)\right)e^{-tV_1(w)}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v.$$

Again using $(a+b)^2 \le 2a^2 + 2b^2$ and the fact that $e^{-tV_1(w)} \le e^{-tr^\pm(w)}$ for any $w \in [0,1]$, we get that the last expression is bounded from above by

$$8c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u \frac{\gamma}{w}tr^-(w)e^{-tr^-(w)}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}} + \frac{\left(\int_v^u \frac{\gamma}{1-w}tr^+(w)e^{-tr^+(w)}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v$$
$$= 16c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u \frac{\gamma}{w}tr^-(w)e^{-tr^-(w)}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v.$$

In the last equality we used a symmetry argument. We can write the last expression as

$$C_\gamma t^{\frac{2-2\gamma}{\gamma}} \iint_{[0,1]^2} \frac{\left(\int_v^u w^{\gamma-2}(tr^-(w))^{\frac{2\gamma-1}{\gamma}} e^{-tr^-(w)}\mathrm{d}w\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v,$$

where $C_\gamma = 16c_\gamma^{\frac{2-\gamma}{\gamma}}\gamma^{\frac{4\gamma-2}{\gamma}}$. Since the function $E_\gamma : [0,\infty) \to [0,\infty)$ defined as $E_\gamma(z) = z^{\frac{2\gamma-1}{\gamma}}e^{-z}$ is bounded from above by $E_\gamma\left(\frac{2\gamma-1}{\gamma}\right)$ we can bound last expression from above by

$$C_\gamma t^{\frac{2-2\gamma}{\gamma}} E_\gamma^2(\tfrac{2\gamma-1}{\gamma}) \iint_{[0,1]^2} \frac{\left(\int_v^u w^{\gamma-2}d\mathrm{w}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v$$
$$= C_\gamma t^{\frac{2-2\gamma}{\gamma}} E_\gamma^2(\tfrac{2\gamma-1}{\gamma})(\gamma-2)^{-2} \iint_{[0,1]^2} \frac{\left(u^{\gamma-1}-v^{\gamma-1}\right)^2}{|u-v|^{\gamma+1}}\mathrm{d}u\mathrm{d}v,$$

which is finite from (7.2) in the proof of Lemma 7.2 of [15]. Thus, we have that

$$8\|e^{-tV_1}\|_{\gamma/2}^2 \lesssim t^{\frac{2-2\gamma}{\gamma}}. \tag{4.15}$$

Therefore, if $g^\infty \in \mathscr{H}^{\gamma/2}$, we conclude that

$$\int_0^T \|\rho_t^\infty\|_{\gamma/2}^2\mathrm{d}t \lesssim T\|\bar\rho^\infty\|_{\gamma/2}^2 + T\left\|g^\infty\right\|_{\gamma/2}^2 + \int_0^T t^{\frac{2-2\gamma}{\gamma}}\mathrm{d}t$$
$$\lesssim T\|\bar\rho^\infty\|_{\gamma/2}^2 + T\left\|g^\infty\right\|_{\gamma/2}^2 + T^{\frac{2-\gamma}{\gamma}},$$

which is finite, since $\gamma < 2$.

For (ii), since ρ^∞ is the solution of (2.12) then it satisfies item (ii) of Definition 2.6. In order to see that ρ^∞ satisfies item $i)$ of Definition 2.6, note that using $(a+b)^2 \le 2a^2 + 2b^2$ we have that

$$\int_0^T\int_0^1 \left(\frac{\left(\alpha-\rho_t^\infty(u)\right)^2}{u^\gamma} + \frac{\left(\beta-\rho_t^\infty(u)\right)^2}{(1-u)^\gamma}\right)\mathrm{d}u\mathrm{d}t$$
$$\le 2T\int_0^1\left(\frac{(\alpha-\bar\rho^\infty(u))^2}{u^\gamma} + \frac{(\beta-\bar\rho^\infty(u))^2}{(1-u)^\gamma}\right)\mathrm{d}u + \frac{8\gamma}{c_\gamma}\int_0^T \|e^{-tV_1}\|_{V_1}^2\mathrm{d}t$$
$$= 2T(\beta-\alpha)^2\int_0^1\left(u^\gamma + (1-u)^\gamma\right)\mathrm{d}u + \frac{8\gamma}{c_\gamma}\int_0^T \|e^{-tV_1}\|_{V_1}^2\mathrm{d}t$$
$$\le 2^\gamma(\beta-\alpha)^2T + \frac{8\gamma}{c_\gamma}\int_0^T \|e^{-tV_1}\|_{V_1}^2\mathrm{d}t.$$

For the term on the right hand side of last expression we first see that we can extend continuously the function e^{-tV_1} in such a way that it vanishes at 0 and at 1. There exists a constant C_2 (see 4.2) such that the previous expression is bounded from above by

$$2^\gamma(\beta-\alpha)^2T + \frac{8\gamma C_2^2}{c_\gamma}\int_0^T \|e^{-tV_1}\|_{\gamma/2}^2\mathrm{d}t. \tag{4.16}$$

Thus, we obtain the desired result by using (4.15). □

Lemma 4.4. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function, such that $\rho_0 - \bar{\rho}^\infty \in \mathscr{H}^{\gamma/2}$. Furthermore, let ρ^κ and ρ^∞ be the weak solutions of (2.10) and (2.12), respectively, and with the same initial condition ρ_0. Let $\hat{\rho}^\kappa_t := \rho^\kappa_{t/\kappa}$, for all $t \in [0,T]$. Then $\hat{\rho}^\kappa$ converges strongly to ρ^∞ in $L^2(0,T;L^2)$, as κ goes to ∞.*

Proof. It is enough to show that

$$\int_0^t \|\hat{\rho}^\kappa_s - \rho^\infty_s\|^2 \,\mathrm{d}s = \int_0^t \|\hat{\varphi}^\kappa_s - \varphi^\infty_s\|^2 \,\mathrm{d}s \lesssim \frac{1}{\sqrt{\kappa}} \tag{4.17}$$

for all $t \in [0,T]$ where $\hat{\varphi}^\kappa_t = \hat{\rho}^\kappa_t - \bar{\rho}^\infty$ and $\varphi^\infty_t = (\rho_0 - \bar{\rho}^\infty)e^{-tV_1}$. It is not difficult to see that $\hat{\varphi}^\kappa_t$ satisfies

$$\begin{aligned}&\langle \hat{\varphi}^\kappa_t, G_t\rangle - \langle \varphi_0, G_0\rangle - \int_0^t \langle \hat{\varphi}^\kappa_s, \partial_s G_s\rangle \,\mathrm{d}s + \int_0^t \langle \hat{\varphi}^\kappa_s, G_s\rangle_{V_1} \mathrm{d}s \\ &\quad - \frac{1}{\kappa}\int_0^t \langle \hat{\rho}^\kappa_s, \mathbb{L} G_s\rangle \mathrm{d}s = 0\end{aligned} \tag{4.18}$$

for all functions $G \in C_c^{1,\infty}([0,T]\times(0,1))$. Then, stating that $\hat{\delta}^k := \hat{\varphi}^\kappa - \varphi^\infty$, we have that

$$\begin{aligned}&\langle \hat{\delta}^\kappa_t, G_t\rangle - \int_0^t \left\langle \hat{\delta}^\kappa_s, \left(\frac{1}{\kappa}\mathbb{L} + \partial_s\right) G_s \right\rangle \mathrm{d}s + \int_0^t \left\langle \hat{\delta}^\kappa_s, G_s \right\rangle_{V_1} \\ &\quad = \frac{1}{\kappa}\int_0^t \langle \rho^\infty_s, G_s\rangle_{\gamma/2} \mathrm{d}s\end{aligned} \tag{4.19}$$

for any function $G \in C_c^{1,\infty}([0,T]\times(0,1))$. Let $\{\hat{H}^\kappa_n\}_{n\geq 1}$, be a sequence of functions in $C_c^{1,\infty}([0,T],(0,1))$ converging to $\hat{\delta}^\kappa$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$. Now, for $n \geq 1$ we define the test function $\hat{G}^\kappa_n(s,u) = \int_s^t \hat{H}^\kappa_n(r,u)\,\mathrm{d}r$. Plugging $\hat{G}^\kappa_n$ into (4.19) and using a similar argument as in proof of Lemma 4.2 we get that

$$\begin{aligned}&\int_0^t \|\hat{\delta}^\kappa_s\|^2 \,\mathrm{d}s + \frac{1}{2\kappa}\left\| \int_0^t \hat{\delta}^\kappa_s \,\mathrm{d}s\right\|^2_{\gamma/2} + \frac{1}{2}\left\| \int_0^t \hat{\delta}^\kappa_s \,\mathrm{d}s\right\|^2_{V_1} \\ &\quad = \frac{1}{\kappa}\int_0^t \left\langle \rho^\infty_s, \int_s^t \hat{\delta}^\kappa_r \,\mathrm{d}r\right\rangle_{\gamma/2} \mathrm{d}s.\end{aligned}$$

By neglecting terms we get that

$$\int_0^t \|\hat{\rho}^\kappa_s - \rho^\infty_s\|^2 \,\mathrm{d}s = \int_0^t \|\hat{\delta}^\kappa_s\|^2 \,\mathrm{d}s \leq \frac{1}{\kappa}\int_0^t \left\langle \rho^\infty_s, \int_s^t \hat{\delta}^\kappa_r \,\mathrm{d}r\right\rangle_{\gamma/2} \mathrm{d}s.$$

Then it is suffices to show that

$$\frac{1}{\kappa}\int_0^t \left\langle \rho^\infty_s, \int_s^t \hat{\delta}^\kappa_r \,\mathrm{d}r\right\rangle_{\gamma/2} \mathrm{d}s \lesssim \frac{1}{\sqrt{\kappa}}.$$

To do this, we start by twice using Cauchy–Schwarz's inequality so that the term at the left hand side of the previous expression is bounded from above by

$$\frac{1}{\kappa}\int_0^t \|\rho_s^\infty\|_{\gamma/2}\left\|\int_s^t \hat\delta_r^\kappa\,\mathrm{d}r\right\|_{\gamma/2}\mathrm{d}s \le \frac{1}{\kappa}\sqrt{\int_0^t \|\rho_s^\infty\|_{\gamma/2}^2\mathrm{d}s}\sqrt{\int_0^t \left\|\int_s^t \hat\delta_r^\kappa\,\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}s}.$$

Since by hypothesis $\rho_0 - \bar\rho^\infty \in \mathscr{H}^{\gamma/2}$ we know from item (i) of Lemma 4.3 that $\rho^\infty \in L^2(0, T; \mathscr{H}^{\gamma/2})$. Thus, from the latter, and by the triangular inequality, the right hand side in the previous expression can be bounded from above by a constant time

$$\frac{1}{\kappa}\sqrt{\int_0^t\left(\int_s^t \|\hat\delta_r^\kappa\|_{\gamma/2}\mathrm{d}r\right)^2\mathrm{d}s} \lesssim \frac{1}{\kappa}\sqrt{t\left(\int_0^t \|\hat\delta_r^\kappa\|_{\gamma/2}\mathrm{d}r\right)^2}.$$

By using Cauchy–Schwarz's inequality again, the term on the right hand side in the last expression is bounded from above by

$$\begin{aligned}\frac{1}{\kappa}\sqrt{t^2\int_0^t \|\hat\delta_r^\kappa\|_{\gamma/2}^2\mathrm{d}r} &= \frac{1}{\kappa}\sqrt{t^2\int_0^t \|\hat\rho_r^\kappa - \rho_r^\infty\|_{\gamma/2}^2\mathrm{d}r}\\ &\lesssim \frac{1}{\kappa}\sqrt{2t^2\int_0^t \|\hat\rho_r^\kappa\|_{\gamma/2}^2 + \|\rho_r^\infty\|_{\gamma/2}^2\mathrm{d}r}.\end{aligned}$$

In the last inequality we used the Minkowski's inequality and the fact that $(a+b)^2 \le 2a^2 + 2b^2$. Now, since $\int_0^t \|\hat\rho_r^\kappa\|_{\gamma/2}^2\mathrm{d}r \lesssim \kappa$ (this is due to item (i) of Theorem 3.2 and a change of variables) and $\rho^\infty \in L^2(0, T; \mathscr{H}^{\gamma/2})$, we can see that

$$\frac{1}{\kappa}\sqrt{2t^2\int_0^t \|\hat\rho_r^\kappa\|_{\gamma/2}^2 + \|\rho_r^\infty\|_{\gamma/2}^2\mathrm{d}r} \lesssim \frac{1}{\kappa}\sqrt{\kappa+1} \lesssim \frac{1}{\sqrt{\kappa}},$$

and we are done. □

4.1. Proof of Item (i) of Theorem 2.13.

Recall φ_t^κ defined in (4.4). Note that it is enough to show (4.9) with $\|\cdot\|$ replaced with $\|\cdot\|_{\gamma/2}$. From (4.10) we obtain, for $\varepsilon > 0$, that

$$\langle \delta_{t+\varepsilon}^\kappa, G_{t+\varepsilon}\rangle - \langle \delta_t^\kappa, G_t\rangle - \int_t^{t+\varepsilon} \langle \delta_s^\kappa, (\mathbb{L} + \partial_s)\, G_s\rangle\,\mathrm{d}s = -\kappa\int_t^{t+\varepsilon} \langle \varphi_s^\kappa, G_s\rangle_{V_1}\mathrm{d}s \tag{4.20}$$

for any function $G \in C_c^{1,\infty}([0, T] \times [0, 1])$. Let $\{H_n^\kappa\}_{n\ge1}$ be a sequence of functions in $C_c^{1,\infty}([0, T], (0, 1))$ converging to δ^κ with respect to the norm of $L^2(0, T; \mathscr{H}_0^{\gamma/2})$ as $n \to \infty$. Now, for $n \ge 1$, we define the test function $G_n^\kappa(u) =$

$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} H_n^\kappa(r,u)\mathrm{d}r$. Plugging G_n^κ into last equality and taking $n \to \infty$, a similar argument to the one of the proof of Lemma 4.2 allows us to get

$$\frac{1}{\varepsilon}\left\langle \delta_{t+\varepsilon}^\kappa - \delta_t^\kappa, \int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle + \varepsilon \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 = \kappa \int_t^{t+\varepsilon} \left\langle \varphi_s^\kappa, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s.$$

Integrating last equality over $[0, \tilde{t}]$, we get

$$\varepsilon \int_0^{\tilde{t}} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 \mathrm{d}t = \kappa \int_0^{\tilde{t}} \int_t^{t+\varepsilon} \left\langle \varphi_s^\kappa, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s\, \mathrm{d}t - \frac{1}{\varepsilon}\int_0^{\tilde{t}} \left\langle \delta_{t+\varepsilon}^\kappa - \delta_t^\kappa, \int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle \mathrm{d}t. \tag{4.21}$$

Now we use Cauchy–Schwarz's inequality, Hardy's inequality and (4.5) to get that

$$\begin{aligned} &\kappa \int_0^{\tilde{t}} \int_t^{t+\varepsilon} \left\langle \varphi_s^\kappa, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle_{V_1} \mathrm{d}s\, \mathrm{d}t \\ &\lesssim \kappa \int_0^{\tilde{t}} \int_t^{t+\varepsilon} \|\varphi_s^\kappa\|_{\gamma/2} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2} \mathrm{d}s\, \mathrm{d}t \\ &\lesssim \kappa \sqrt{\int_0^{\tilde{t}} \int_t^{t+\varepsilon} \|\varphi_s^\kappa\|_{\gamma/2}^2 \mathrm{d}s \mathrm{d}t} \sqrt{\int_0^{\tilde{t}} \int_t^{t+\varepsilon} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 \mathrm{d}s\, \mathrm{d}t} \\ &\lesssim \kappa \varepsilon \sqrt{\tilde{t}} \sqrt{\int_0^{\tilde{t}} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\|_{\gamma/2}^2 \mathrm{d}t}. \end{aligned} \tag{4.22}$$

Let us estimate the second term on the right hand side (4.21). First note that by changing variables we have that

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_0^{\tilde{t}} \left\langle \delta_{t+\varepsilon}^\kappa - \delta_t^\kappa, \int_t^{t+\varepsilon} \delta_r^\kappa \mathrm{d}r \right\rangle \mathrm{d}t \\ &= \frac{1}{\varepsilon}\int_0^{\tilde{t}} \int_t^{t+\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}r \mathrm{d}t - \frac{1}{\varepsilon}\int_0^{\tilde{t}} \int_t^{t+\varepsilon} \langle \delta_{t+\varepsilon}^\kappa, \delta_r^\kappa \rangle \mathrm{d}r \mathrm{d}t \\ &= \frac{1}{\varepsilon}\int_0^{\tilde{t}} \int_r^{r+\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}t \mathrm{d}r - \frac{1}{\varepsilon}\int_\varepsilon^{\tilde{t}+\varepsilon} \int_{t-\varepsilon}^{t} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}r \mathrm{d}t. \end{aligned} \tag{4.23}$$

The term $\frac{1}{\varepsilon}\int_0^{\tilde{t}}\int_r^{r+\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}t \mathrm{d}r$ can be split as

$$\frac{1}{\varepsilon}\left(\int_0^{\varepsilon} \int_r^{\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}t \mathrm{d}r + \int_0^{\varepsilon} \int_\varepsilon^{r+\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}t \mathrm{d}r + \int_\varepsilon^{\tilde{t}} \int_r^{r+\varepsilon} \langle \delta_t^\kappa, \delta_r^\kappa \rangle \mathrm{d}t \mathrm{d}r \right).$$

By Fubini's theorem, we have that the term $\frac{1}{\varepsilon}\int_{\varepsilon}^{\tilde{t}+\varepsilon}\int_{t-\varepsilon}^{t}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}r\,\mathrm{d}t$, which appears in (4.23), is equal to

$$\frac{1}{\varepsilon}\left(\int_0^{\varepsilon}\int_{\varepsilon}^{r+\varepsilon}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}t\,\mathrm{d}r+\int_{\varepsilon}^{\tilde{t}}\int_{r}^{r+\varepsilon}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}t\,\mathrm{d}r+\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\int_{r}^{\tilde{t}+\varepsilon}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}t\,\mathrm{d}r\right).$$

Therefore we can write the second term on the right hand side of (4.21) as

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\int_{r}^{\tilde{t}+\varepsilon}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}t\,\mathrm{d}r+\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_r^{\varepsilon}\langle\delta_t^\kappa,\delta_r^\kappa\rangle \mathrm{d}t\,\mathrm{d}r\\
&\leq\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\delta_t^\kappa\|\|\delta_r^\kappa\|\mathrm{d}t\,\mathrm{d}r+\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_0^{\varepsilon}\|\delta_t^\kappa\|\|\delta_r^\kappa\|\mathrm{d}t\,\mathrm{d}r\\
&=\frac{1}{\varepsilon}\left(\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\delta_t^\kappa\|\,\mathrm{d}t\right)^2+\frac{1}{\varepsilon}\left(\int_0^{\varepsilon}\|\delta_t^\kappa\|\,\mathrm{d}t\right)^2\\
&\leq\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\delta_t^\kappa\|^2\mathrm{d}t+\int_0^{\varepsilon}\|\delta_t^\kappa\|^2\mathrm{d}t,
\end{aligned}\tag{4.24}$$

where in the inequalities above we used Cauchy–Schwarz's inequality. Then, using (4.22) and (4.24) in (4.21), we obtain that

$$\begin{aligned}
\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\delta_r^\kappa\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}t\lesssim\kappa\sqrt{\tilde{t}}\sqrt{\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\delta_r^\kappa\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}t}\\
+\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\delta_t^\kappa\|^2\mathrm{d}t+\frac{1}{\varepsilon}\int_0^{\varepsilon}\|\delta_t^\kappa\|^2\mathrm{d}t.
\end{aligned}\tag{4.25}$$

Taking $\varepsilon\to 0$, using Lebesgue's differentiation theorem (see Theorem 1.35 in [22]) and the fact that $\delta_0^\kappa=0$ (since the initial condition for ρ^κ and ρ^0 is the same) we get that

$$\int_0^{\tilde{t}}\|\delta_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\lesssim\kappa\sqrt{\tilde{t}}\sqrt{\int_0^{\tilde{t}}\|\delta_t^\kappa\|_{\gamma/2}^2\mathrm{d}t}+\|\delta_{\tilde{t}}^\kappa\|^2,$$

for all $\tilde{t}\in[0,T]$. Integrating the last inequality over $[0,T]$ and using Cauchy–Schwarz's inequality and (4.9), we conclude that

$$\int_0^T\int_0^{\tilde{t}}\|\delta_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\,\mathrm{d}\tilde{t}\lesssim\kappa T\sqrt{\int_0^T\int_0^{\tilde{t}}\|\delta_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\,\mathrm{d}\tilde{t}}+\kappa T^2.\tag{4.26}$$

Then, by a simple computation, we have that

$$\int_0^T\int_0^{\tilde{t}}\|\delta_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\,\mathrm{d}\tilde{t}\lesssim\kappa T^2.\tag{4.27}$$

By Fubini's theorem, we get that

$$\int_0^T \int_0^{\tilde{t}} \|\delta_t^\kappa\|_{\gamma/2}^2 \, \mathrm{d}t \mathrm{d}\tilde{t} = \int_0^T (T-t)\|\delta_t^\kappa\|_{\gamma/2}^2 \, \mathrm{d}t \geq \frac{T}{2}\int_0^{T/2} \|\delta_t^\kappa\|_{\gamma/2}^2 \, \mathrm{d}t. \tag{4.28}$$

The result now follows from (4.27) and (4.28). □

4.2. Proof of Item (ii) of Theorem 2.13

Recall $\hat{\varphi}_t^\kappa$ and φ_t^∞ defined in Lemma 4.4. It is enough to show (4.17) with $\|\cdot\|$ replaced with $\|\cdot\|_{V_1}$:

$$\int_0^T \|\hat{\varphi}_t^\kappa - \varphi_t^\infty\|_{V_1}^2 \, \mathrm{d}t \lesssim \frac{1}{\sqrt{\kappa}}. \tag{4.29}$$

From (4.19), we obtain, for $\varepsilon > 0$, that

$$\begin{aligned} \langle \hat{\delta}_{t+\varepsilon}^\kappa, G_{t+\varepsilon}\rangle - \langle \hat{\delta}_t^\kappa, G_t\rangle - \int_t^{t+\varepsilon} \langle \hat{\delta}_s^\kappa, \Big(\frac{1}{\kappa}\mathbb{L} + \partial_s\Big) G_s\rangle \, \mathrm{d}s \\ + \int_t^{t+\varepsilon} \langle \hat{\delta}_s^\kappa, G_s\rangle_{V_1} \, \mathrm{d}s = \frac{1}{\kappa}\int_t^{t+\varepsilon} \langle \rho_s^\infty, G_s\rangle_{\gamma/2} \mathrm{d}s \end{aligned} \tag{4.30}$$

for any function $G \in C_c^{1,\infty}([0,T]\times[0,1])$. Let $\{\hat{H}_n^\kappa\}_{n\geq 1}$ be a sequence of functions in $C_c^{1,\infty}([0,T],(0,1))$ converging to $\hat{\delta}^\kappa$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$ as $n\to\infty$. Now, for $n\geq 1$ we define the test functions $\hat{G}_n^\kappa(u) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{H}_n^\kappa(r,u)\mathrm{d}r$. Plugging $\hat{G}_n^\kappa$ into (4.30) and taking $n\to\infty$, a similar argument to the one of the proof of Lemma 4.2 allows us to get

$$\begin{aligned} \frac{1}{\varepsilon}\Big\langle \hat{\delta}_{t+\varepsilon}^\kappa - \hat{\delta}_t^\kappa, \int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\rangle + \frac{\varepsilon}{\kappa}\Big\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\|_{\gamma/2}^2 \\ + \varepsilon \Big\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\|_{V_1}^2 = \frac{1}{\kappa}\int_t^{t+\varepsilon} \Big\langle \rho_s^\infty, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\rangle_{\gamma/2} \mathrm{d}s. \end{aligned} \tag{4.31}$$

By neglecting the term $\frac{\varepsilon}{\kappa}\Big\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\|_{\gamma/2}^2$ in (4.31) and then integrating over $[0,\tilde{t}]$ we get that

$$\begin{aligned} \varepsilon \int_0^{\tilde{t}} \Big\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\|_{V_1}^2 \mathrm{d}t \leq \frac{1}{\kappa}\int_0^{\tilde{t}}\int_t^{t+\varepsilon} \Big\langle \rho_s^\infty, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\rangle_{\gamma/2} \mathrm{d}s \, \mathrm{d}t \\ - \frac{1}{\varepsilon}\int_0^{\tilde{t}} \Big\langle \hat{\delta}_{t+\varepsilon}^\kappa - \hat{\delta}_t^\kappa, \int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\rangle \mathrm{d}t. \end{aligned} \tag{4.32}$$

Now we use Cauchy–Schwarz's inequality twice in order to get that the first term on the right hand side in the previous expression is bounded from above by

$$\frac{1}{\kappa}\int_0^{\tilde{t}}\int_t^{t+\varepsilon} \|\rho_s^\infty\|_{\gamma/2} \Big\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa \, \mathrm{d}r \Big\|_{\gamma/2} \mathrm{d}s \, \mathrm{d}t$$

$$\leq \frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2\mathrm{d}s\mathrm{d}t}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}s,\mathrm{d}t}$$

$$\leq \frac{\sqrt{\varepsilon}}{\kappa}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2\mathrm{d}s\mathrm{d}t}\sqrt{\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}t}. \tag{4.33}$$

By a similar argument as to the one in the proof of item i) of Theorem 2.13, we have that the second term on the right hand side in (4.32) is bounded from above by

$$\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\hat{\delta}_t^\kappa\|^2\mathrm{d}t + \frac{1}{\varepsilon}\int_0^{\varepsilon}\|\hat{\delta}_t^\kappa\|^2\mathrm{d}t. \tag{4.34}$$

Therefore, by using (4.33) and (4.34) in (4.32), we get that

$$\begin{aligned}\int_0^{\tilde{t}}&\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa\mathrm{d}r\right\|_{V_1}^2\mathrm{d}t\\ &\leq \frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2\mathrm{d}s\mathrm{d}t}\sqrt{\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa\mathrm{d}r\right\|_{\gamma/2}^2\mathrm{d}t}\\ &\quad+\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\hat{\delta}_t^\kappa\|^2\mathrm{d}t+\frac{1}{\varepsilon}\int_0^{\varepsilon}\|\hat{\delta}_t^\kappa\|^2\mathrm{d}t.\end{aligned} \tag{4.35}$$

Taking $\varepsilon \to 0$, using Lebesgue's differentiation theorem (see Theorem 1.35 in [22]) and the fact that $\hat{\delta}_0^\kappa = 0$ we get that

$$\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{V_1}^2\mathrm{d}t \leq \frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\|\rho_t^\infty\|_{\gamma/2}^2\mathrm{d}t}\sqrt{\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{\gamma/2}^2\mathrm{d}t}+\|\hat{\delta}_{\tilde{t}}^\kappa\|^2$$

for all $\tilde{t} \in [0, T]$. Integrating the previous expression over $[0, T]$ and using the Cauchy–Schwarz's inequality we get that

$$\begin{aligned}\int_0^T\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{V_1}^2\mathrm{d}t\mathrm{d}\tilde{t} &\leq \frac{1}{\kappa}\sqrt{\int_0^T\int_0^{\tilde{t}}\|\rho_t^\infty\|_{\gamma/2}^2\mathrm{d}t\mathrm{d}\tilde{t}}\sqrt{\int_0^T\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\mathrm{d}\tilde{t}}\\ &\quad+\int_0^T\|\hat{\delta}_{\tilde{t}}^\kappa\|^2\mathrm{d}\tilde{t}\\ &\lesssim \frac{1}{\kappa}\sqrt{\int_0^T\int_0^T\|\hat{\delta}_t^\kappa\|_{\gamma/2}^2\mathrm{d}t\mathrm{d}\tilde{t}}+\frac{1}{\sqrt{\kappa}},\\ &\lesssim \frac{1}{\kappa}\sqrt{2T\int_0^T\|\hat{\rho}_t^\kappa\|_{\gamma/2}^2+\|\rho_t^\infty\|_{\gamma/2}^2\mathrm{d}t}+\frac{1}{\sqrt{\kappa}},\\ &\lesssim \frac{1}{\kappa}\sqrt{(\kappa+2)}+\frac{1}{\sqrt{\kappa}}.\end{aligned} \tag{4.36}$$

In the second inequality above we used the fact that $\rho^\infty \in L^2(0,T;\mathscr{H}^{\gamma/2})$ (see item i) of Lemma 4.3) and (4.29), while in the third inequality of we used Minkoski's inequality and the fact that $(a+b)^2 \le 2a^2+2b^2$. Finally, the last inequality of (4.36) is true, since $\rho^\infty \in L^2(0,T;\mathscr{H}^{\gamma/2})$ and item i) of Theorem 3.2.

Then, by a simple computation, we have that

$$\int_0^T \int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{V_1}^2 \mathrm{d}t\mathrm{d}\tilde{t} \lesssim \frac{1}{\sqrt{\kappa}}. \tag{4.37}$$

By Fubini's theorem, we have that

$$\int_0^T \int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{V_1}^2 \,\mathrm{d}t\mathrm{d}\tilde{t} = \int_0^T (T-t)\|\hat{\delta}_t^\kappa\|_{V_1}^2 \,\mathrm{d}t \ge \frac{T}{2}\int_0^{T/2} \|\hat{\delta}_t^\kappa\|_{V_1}^2 \,\mathrm{d}t. \tag{4.38}$$

The result now follows from (4.37) and (4.38). □

5. Proof of Theorem 2.15

In this section we prove items (i) and (ii) of Theorem 2.15. Now we are interested in analyzing the convergence of the stationary solution $\bar{\rho}^\kappa$ as $\kappa \to 0$ and $\kappa \to \infty$. From Definition 2.9, for $\kappa \ge 0$, and for $\bar{\varphi}^\kappa = \bar{\rho}^\kappa - \bar{\rho}^\infty$ we have that $\bar{\varphi}^\kappa \in \mathscr{H}_0^{\gamma/2}$ and

$$\langle \bar{\varphi}^\kappa, -\mathbb{L}G\rangle + \kappa\langle \bar{\varphi}^\kappa, G\rangle_{V_1} = I_{\bar{\rho}^\infty}(G) \tag{5.1}$$

for any test function G of compact support included in $(0,1)$. Above $I_{\bar{\rho}^\infty} : \mathscr{H}_0^{\gamma/2} \to \mathbb{R}$ is a linear form defined by $I_{\bar{\rho}^\infty}(G) = \langle \bar{\rho}^\infty, \mathbb{L}G\rangle$. Moreover, this linear form is continuous. Indeed, using integration by parts given in Proposition 3.3 in [15] we have that

$$\begin{aligned} |I_{\bar{\rho}^\infty}(G)| &= \left|\int_0^1 \bar{\rho}^\infty(u)\mathbb{L}G(u)\mathrm{d}u\right| \\ &= \frac{c_\gamma}{2}\left|\iint_{[0,1]^2} \frac{(\bar{\rho}^\infty(u)-\bar{\rho}^\infty(v))(G(u)-G(v))}{|u-v|^{\gamma+1}}\mathrm{d}v\mathrm{d}u\right| \\ &\le \|\bar{\rho}^\infty\|_{\gamma/2}\|G\|_{\gamma/2} < \infty. \end{aligned} \tag{5.2}$$

Above we used Cauchy–Schwarz's inequality and the fact that $\|\bar{\rho}^\infty\|_{\gamma/2}$ is finite (see (4.1)). Therefore, $|I_{\rho^\infty}(G)| \lesssim \|G\|_{\mathscr{H}_0^{\gamma/2}}$.

Then it is enough to analyze the behavior of $\bar{\varphi}^\kappa$. We claim that we can take $G = \bar{\varphi}^\kappa$ in (5.1). The justification is postponed to the end of the proof. Whence, from (5.2), we have that

$$\|\bar{\varphi}^\kappa\|_{\gamma/2}^2 + \kappa\|\bar{\varphi}^\kappa\|_{V_1}^2 = I_{\bar{\rho}^\infty}(\bar{\varphi}^\kappa) \lesssim \|\bar{\varphi}^\kappa\|_{\gamma/2}, \tag{5.3}$$

from which we conclude that $\|\bar{\varphi}^\kappa\|_{\gamma/2} < \infty$. Plugging this back into (5.3) we get that

$$\|\bar{\varphi}^\kappa\|_{V_1} \lesssim \frac{1}{\sqrt{\kappa}}. \tag{5.4}$$

Now, note that $\bar{\varphi}^0 \in \mathscr{H}_0^{\gamma/2}$ satisfies $\langle \bar{\varphi}^0, -\mathbb{L}G\rangle = I_{\bar{\rho}^\infty}(G)$, for any function $G \in C_c^\infty((0,1))$. Then $\bar{\varphi}^\kappa - \bar{\varphi}^0$ satisfies

$$\langle \bar{\varphi}^\kappa - \bar{\varphi}^0, -\mathbb{L}G\rangle + \kappa \langle \bar{\varphi}^\kappa, G\rangle_{V_1} = 0$$

for any function $G \in C_c^\infty((0,1))$. We claim that we can take $G = \bar{\varphi}^\kappa - \bar{\varphi}^0$ in the previous equality. The proof is analogous to the one done at the end of this section. Thus, we get that

$$\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{\gamma/2}^2 = \kappa \langle \bar{\varphi}^\kappa, \bar{\varphi}^0 - \bar{\varphi}^\kappa\rangle_{V_1} \le \kappa \|\bar{\varphi}^\kappa\|_{V_1} \|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{V_1}.$$

From (5.4) and fractional Hardy's inequality given in (4.2) we have that

$$\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{\gamma/2}^2 \lesssim \sqrt{\kappa}\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{V_1} \lesssim \sqrt{\kappa}\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{\gamma/2},$$

from which we conclude that $\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{\gamma/2} \lesssim \sqrt{\kappa}$. Then $\bar{\varphi}^\kappa$ converges to $\bar{\varphi}^0$, as $\kappa \to 0$ in the $\|\cdot\|_{\gamma/2}$ norm. So far we have proved item (i).

Remark 5.1. From the fractional Hardy's inequality (see 4.2) the convergence is also true in $L^2_{V_1}$, and since

$$\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|_{V_1} \ge V_1(\tfrac{1}{2})\|\bar{\varphi}^\kappa - \bar{\varphi}^0\|,$$

we conclude that the convergence also holds in L^2.

For item (ii), by (5.4) we get that $\|\bar{\varphi}^\kappa\|_{V_1} \to 0$, and so $\|\bar{\varphi}^\kappa\| \to 0$ as $k \to \infty$.

We conclude this proof by showing that we can take $G = \bar{\varphi}^\kappa$ in (5.1). Indeed, since $C_c^\infty((0,1))$ is dense in $\mathscr{H}_0^{\gamma/2}$, there exists a sequence $\{\bar{H}_n^\kappa\}_{n\ge 1}$ in $C_c^\infty((0,1))$ converging to $\bar{\varphi}^\kappa$, i.e, $\|\bar{H}_n^\kappa - \bar{\varphi}^\kappa\|_{\gamma/2} \to 0$ as $n \to \infty$. Observe that as a result of the latter and (4.2) we also have $\|\bar{H}_n^\kappa - \bar{\varphi}^\kappa\|_{V_1} \to 0$ as $n \to \infty$. Using Cauchy–Schwarz's inequality we have that

$$\begin{aligned}
\langle \bar{\varphi}^\kappa, \bar{H}_n^\kappa - \bar{\varphi}^\kappa\rangle_{\gamma/2} &\le \|\bar{\varphi}^\kappa\|_{\gamma/2}\|\bar{H}_n^\kappa - \bar{\varphi}^\kappa\|_{\gamma/2},\\
\langle \bar{\varphi}^\kappa, \bar{H}_n^\kappa - \bar{\varphi}^\kappa\rangle_{V_1} &\le \|\bar{\varphi}^\kappa\|_{V_1}\|\bar{H}_n^\kappa - \bar{\varphi}^\kappa\|_{V_1},\\
I_{\bar{\rho}^\infty}(\bar{H}_n^\kappa - \bar{\varphi}^\kappa) &\le \|\bar{\rho}^\infty\|_{\gamma/2}\|\bar{H}_n^\kappa - \bar{\varphi}^\kappa\|_{\gamma/2},
\end{aligned}$$

all going to 0 as $n \to \infty$. Thus, we can rewrite (5.1) as

$$\begin{aligned}
&\langle \bar{\varphi}^\kappa, -\mathbb{L}\bar{\varphi}^\kappa\rangle + \langle \bar{\varphi}^\kappa, -\mathbb{L}(\bar{H}_n^\kappa - \bar{\varphi}^\kappa)\rangle + \kappa(\langle \bar{\varphi}^\kappa, \bar{\varphi}^\kappa\rangle_{V_1} + \langle \bar{\varphi}^\kappa, \bar{H}_n^\kappa - \bar{\varphi}^\kappa\rangle_{V_1})\\
&\quad = I_{\bar{\rho}^\infty}(\bar{\varphi}^\kappa) + I_{\bar{\rho}^\infty}(\bar{H}_n^\kappa - \bar{\varphi}^\kappa).
\end{aligned}$$

Now it is enough to take $n \to \infty$. □

6. Uniqueness of Weak Solutions

In this section we prove Lemmas 2.8 and 2.11. For Lemma 2.8, we only focus in the proof of the uniqueness for the weak solutions of (2.10) for $\hat{\kappa} = \kappa > 0$. The proof of the uniqueness of the weak solutions of (2.10) for $\kappa = 0$ and (2.12) is analogous, the difference is that only the first two items in Lemma 6.1 below are required. Finally, in Section 6.2 we prove Lemma 2.11.

6.1. Proof of Lemma 2.15

Let $\rho^{\kappa,1}$ and $\rho^{\kappa,2}$ two weak solutions of (2.10) with the same initial condition and let us denote $\tilde{\rho}^\kappa = \rho^{\kappa,1} - \rho^{\kappa,2}$. For almost every $t \in [0, T]$, we identify $\tilde{\rho}^\kappa_t$ with its continuous representation on $[0, 1]$. Therefore, by Remark 2.4 we have $\tilde{\rho}^\kappa_t(0) = \tilde{\rho}^\kappa_t(1) = 0$. Since $\mathscr{H}_0^{\gamma/2}$ is equal to the set of functions in $\mathscr{H}^{\gamma/2}$ vanishing at 0 and 1 we have that $\tilde{\rho}^\kappa_t \in \mathscr{H}_0^{\gamma/2}$ for a.e. time $t \in [0, T]$ and, in fact, $\tilde{\rho}^\kappa \in L^2(0, T; \mathscr{H}_0^{\gamma/2})$. Moreover, for any $t \in [0, T]$ and all functions $G \in C_c^{1,\infty}([0, T] \times (0, 1))$ we have

$$\langle \tilde{\rho}^\kappa_t, G_t \rangle - \int_0^t \Big\langle \tilde{\rho}^\kappa_s, \Big(\partial_s + \mathbb{L}\Big) G_s \Big\rangle \mathrm{d}s + \kappa \int_0^t \langle \tilde{\rho}^\kappa_s, G_s \rangle_{V_1} \mathrm{d}s = 0. \tag{6.1}$$

Note that, it is easy to show that $C_c^{1,\infty}([0, T] \times (0, 1))$ is dense in $L^2(0, T; \mathscr{H}_0^{\gamma/2})$. Let $\{H^\kappa_n\}_{n\geq 1}$ be a sequence of functions in $C_c^{1,\infty}([0, T] \times (0, 1))$ converging to $\tilde{\rho}^\kappa$ with respect to the norm of $L^2(0, T; \mathscr{H}_0^{1/2})$ as $n \to \infty$. For $n \geq 1$, we define the test functions $\forall t \in [0, T]$, $\forall u \in [0, 1]$, $G^\kappa_n(t, u) = \int_t^T H^\kappa_n(s, u)\,\mathrm{d}s$. Plugging G^κ_n into (6.1) and letting $n \to \infty$ we conclude by Lemma 6.1 below that

$$\int_0^T \|\tilde{\rho}^\kappa_s\|^2 \mathrm{d}s + \frac{1}{2} \Big\| \int_0^T \tilde{\rho}^\kappa_s \mathrm{d}s \Big\|^2_{\gamma/2} + \frac{\kappa}{2} \Big\| \int_0^T \tilde{\rho}^\kappa_s \mathrm{d}s \Big\|^2_{V_1} = 0. \tag{6.2}$$

Recall that $\langle \cdot, \cdot \rangle_{V_1}$ (resp. $\| \cdot \|_{V_1}$) is the scalar product (resp. the norm) corresponding to the Hilbert space $L^2_{V_1}$.

Then, it follows that for almost every time $s \in [0, T]$ the continuous function $\tilde{\rho}^\kappa_s$ is equal to 0 and we conclude the uniqueness of the weak solutions to (2.10).

Lemma 6.1. *Let $\{G^\kappa_n\}_{n\geq 1}$ be defined as above. We have*

(i) $\displaystyle \lim_{n\to\infty} \int_0^T \big\langle \tilde{\rho}^\kappa_s, (\partial_s G^\kappa_n)(s, \cdot) \big\rangle\, ds = - \int_0^T \|\tilde{\rho}^\kappa_s\|^2 ds;$

(ii) $\displaystyle \lim_{n\to\infty} \int_0^T \big\langle \tilde{\rho}^\kappa_s, \mathbb{L} G^\kappa_n(s, \cdot) \big\rangle\, ds = -\frac{1}{2} \Big\| \int_0^T \tilde{\rho}^\kappa_s ds \Big\|^2_{\gamma/2};$

(iii) $\displaystyle \lim_{n\to\infty} \int_0^T \big\langle \tilde{\rho}^\kappa_s, G^\kappa_n(s, \cdot) \big\rangle_{V_1} ds = \frac{1}{2} \Big\| \int_0^T \tilde{\rho}^\kappa_s ds \Big\|^2_{V_1} < \infty.$

Proof. The proof of this lemma is quite similar to the proof of items (i), (ii) and (iii) in the proof of Lemma 4.2. For that reason we just sketch the main steps of the proof and we leave the details to the reader. For (i) we have that

$$- \int_0^T \big\langle \tilde{\rho}^\kappa_s, (\partial_s G^\kappa_n)(s, \cdot) \big\rangle \mathrm{d}s = \int_0^T \big\langle \tilde{\rho}^\kappa_s, H^\kappa_n(s, \cdot) - \tilde{\rho}^\kappa_s \big\rangle \mathrm{d}s + \int_0^T \|\tilde{\rho}^\kappa_s\|^2 \mathrm{d}s, \tag{6.3}$$

and by the Cauchy–Schwarz inequality,

$$\Big| \int_0^T \big\langle \tilde{\rho}^\kappa_s\,,\ H^\kappa_n(s, \cdot) - \tilde{\rho}^\kappa_s \big\rangle \mathrm{d}s \Big| \leq \sqrt{\int_0^T \|\tilde{\rho}^\kappa_s\|^2 \,\mathrm{d}s} \sqrt{\int_0^T \|H^\kappa_n(s, \cdot) - \tilde{\rho}^\kappa_s\|^2 \,\mathrm{d}s}, \tag{6.4}$$

which goes to 0 as $n \to \infty$.

For (ii), we first use the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [15]) to get

$$\int_0^T \langle \tilde{\rho}_s^\kappa, \mathbb{L}G_n^\kappa(s,\cdot)\rangle \, \mathrm{d}s = -\int_0^T \left\langle \tilde{\rho}_s^\kappa, G_n^\kappa(s,\cdot)\right\rangle_{\gamma/2} \mathrm{d}s,$$

and as in (ii) in the proof of Lemma 4.2 we have that

$$\begin{aligned}\int_0^T \left\langle \tilde{\rho}_s^\kappa, G_n^\kappa(s,\cdot)\right\rangle_{\gamma/2} \mathrm{d}s &= \frac{1}{2}\left\| \int_0^T \tilde{\rho}_s^\kappa \mathrm{d}s \right\|_{\gamma/2}^2 \\ &+ \int_0^T \left\langle \tilde{\rho}_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot) - \tilde{\rho}_t^\kappa\right) \mathrm{d}t \right\rangle_{\gamma/2} \mathrm{d}s.\end{aligned}$$

Now, note that the term on the right hand side of last expression vanishes as $n \to \infty$ as a consequence of a successive use of Cauchy–Schwarz's inequalities. The proof of (iii) is similar to the proof of (ii) by using the fractional Hardy's inequality (see (4.2)) and since $C_c^\infty((0,1))$ is dense in $H_0^{\gamma/2}$ we have that any $g \in H_0^{\gamma/2}$ is also in the space $L^2_{V_1}$ and that (4.2) remains valid for g. In particular, we have that the right hand side of (iii) is finite. We have

$$\begin{aligned}\int_0^T \left\langle \tilde{\rho}_s^\kappa, G_n^\kappa(s,\cdot)\right\rangle_{V_1} \mathrm{d}s &= \frac{1}{2}\left\| \int_0^T \tilde{\rho}_s^\kappa \mathrm{d}s \right\|_{V_1}^2 \\ &+ \int_0^T \left\langle \tilde{\rho}_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot) - \tilde{\rho}_t^\kappa\right) \mathrm{d}t \right\rangle_{V_1} \mathrm{d}s.\end{aligned} \tag{6.5}$$

To conclude the proof of (iii) it is sufficient to prove that the term on the right hand side of last expression vanishes as $n \to \infty$. But this is a consequence of a successive use of the Cauchy–Schwarz inequalities and Hardy's inequality, from which we get

$$\begin{aligned}&\left| \int_0^T \left\langle \tilde{\rho}_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot) - \tilde{\rho}_t^\kappa\right) \mathrm{d}t \right\rangle_{V_1} \mathrm{d}s \right| \\ &\leq CT \sqrt{\int_0^T \left\| \tilde{\rho}_s^\kappa \right\|_{\gamma/2}^2 \mathrm{d}s} \sqrt{\int_0^T \left\| H_n^\kappa(t,\cdot) - \tilde{\rho}_t^\kappa \right\|_{\gamma/2}^2 \mathrm{d}t} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

The proof of the uniqueness of the weak solutions of (2.10) for $\kappa = 0$ is analogous, the difference is that only the first two items in Lemma 6.1 above are required. The uniqueness of the weak solutions of (2.12) is analogous as well, in this case only items (i) and (iii) in Lemma 6.1 above are required. □

6.2. *Proof of Lemma 2.11*

Recall (5.1). As we will see below, by Lax–Milgram's Theorem (see [6]), there exists a unique function $\bar{\varphi}^{\hat{\kappa}} \in \mathcal{H}_0^{\gamma/2}$ which is solution of (5.1). Then, it is not difficult to see that $\bar{\rho}^{\hat{\kappa}} := \bar{\varphi}^{\hat{\kappa}} + \bar{\rho}^\infty$ is the desired weak solution of (2.14). For that purpose, let $a^{\hat{\kappa}} : \mathscr{H}_0^{\gamma/2} \times \mathscr{H}_0^{\gamma/2} \to \mathbb{R}$ be the bilinear form defined, for $G, F \in \mathscr{H}_0^{\gamma/2}$, as

$$a^{\hat{\kappa}}(F, G) = \langle F, G\rangle_{\gamma/2} + \hat{\kappa}\langle F, G\rangle_{V_1}. \tag{6.6}$$

From Lax–Milgram Theorem, in order to conclude the existence and uniqueness it is enough to prove that $a^{\hat{\kappa}}$ is coercive and continuous. For $\hat{\kappa} > 0$, we can easily see that

$$a^{\hat{\kappa}}(G, G) \geq \min\{1, \hat{\kappa}V_1(\tfrac{1}{2})\}\left(\|G\|^2_{\gamma/2} + \|G\|^2\right) = \min\{1, \hat{\kappa}V_1(\tfrac{1}{2})\}\|G\|^2_{\mathscr{H}_0^{\gamma/2}}.$$

For $\hat{\kappa} = 0$, since on $\mathscr{H}_0^{\gamma}$ the norms $\|\cdot\|_{\gamma/2}$ and $\|\cdot\|_{\mathscr{H}^{\gamma/2}}$ are equivalent we have that

$$a^0(G, G) = \|G\|^2_{\gamma/2} \gtrsim \|G\|^2_{\mathscr{H}_0^{\gamma/2}}.$$

Therefore $a^{\hat{\kappa}}$ is coercive for $\hat{\kappa} \geq 0$. Moreover, by using the Cauchy–Schwarz's inequality we obtain that

$$|a^{\hat{\kappa}}(F, G)| \leq \|F\|_{\gamma/2}\|G\|_{\gamma/2} + \hat{\kappa}(\|F\|_{V_1}\|G\|_{V_1}).$$

From the fractional Hardy's inequality (see (4.2)) we have that

$$|a^{\hat{\kappa}}(F, G)| \lesssim (\hat{\kappa} + 1)(\|F\|_{\gamma/2}\|G\|_{\gamma/2})$$

and since on $\mathscr{H}_0^{\gamma/2}$ the norms $\|\cdot\|_{\gamma/2}$ and $\|\cdot\|_{\mathscr{H}^{\gamma/2}}$ are equivalent, we conclude that the bilinear form $a^{\hat{\kappa}}$ is continuous for $\kappa\hat{}\geq 0$. This ends the proof.

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Appendix A: Computations Involving the Generator

Lemma A.1. *For any $x \neq y \in \Lambda_N$, we have*

(i) $L_N^0(\eta_x\eta_y) = \eta_x L_N^0\eta_y + \eta_y L_N^0\eta_x - p(y-x)(\eta_y - \eta_x)^2$,
(ii) $L_N^r(\eta_x\eta_y) = \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x$,
(iii) $L_N^\ell(\eta_x\eta_y) = \eta_x L_N^\ell\eta_y + \eta_y L_N^\ell\eta_x$.

Proof. For (i) we have, by definition of L_N^0, that

$$
\begin{aligned}
L_N^0(\eta_x\eta_y) &= \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\Big[(\sigma^{\bar{x},\bar{y}}\eta)_x(\sigma^{\bar{x},\bar{y}}\eta)_y - \eta_x\eta_y\Big]\\
&= \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\Big[((\sigma^{\bar{x},\bar{y}}\eta)_x\eta_y - \eta_x\eta_y) + ((\sigma^{\bar{x},\bar{y}}\eta)_y\eta_x - \eta_x\eta_y)\\
&\quad + (\sigma^{\bar{x},\bar{y}}\eta)_x(\sigma^{\bar{x},\bar{y}}\eta)_y - (\sigma^{\bar{x},\bar{y}}\eta)_x\eta_y - (\sigma^{\bar{x},\bar{y}}\eta)_y\eta_x + \eta_x\eta_y\Big]\\
&= \eta_x L_N^0\eta_y\\
&\quad + \eta_y L_N^0\eta_x + \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\Big[(\sigma^{\bar{x},\bar{y}}\eta)_x - \eta_x\Big]\Big[(\sigma^{\bar{x},\bar{y}}\eta)_y - \eta_y\Big]\\
&= \eta_x L_N^0\eta_y + \eta_y L_N^0\eta_x - p(y-x)(\eta_y - \eta_x)^2.
\end{aligned}
$$

In order to prove (ii), note that $\Big[(\sigma^{\bar{x}}\eta)_x - \eta_x\Big]\Big[(\sigma^{\bar{x}}\eta)_y - \eta_y\Big]$ is equal to zero, for all $\bar{x} \in \mathbb{Z}$. Thus, by definition of L_N^r, we have that

$$
\begin{aligned}
L_N^r(\eta_x\eta_y) &= \sum_{\bar{x}\in\Lambda_N,\bar{y}\geq N} p(\bar{y}-\bar{x})\,[\eta_{\bar{x}}(1-\beta) + (1-\eta_{\bar{x}})\beta]\Big[(\sigma^{\bar{x}}\eta)_x(\sigma^{\bar{x}}\eta)_y - \eta_x\eta_y\Big]\\
&= \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x\\
&\quad + \sum_{\bar{x}\in\Lambda_N,\bar{y}\geq N} p(\bar{y}-\bar{x})\\
&\qquad [\eta_{\bar{x}}(1-\beta) + (1-\eta_{\bar{x}})\beta]\Big[(\sigma^{\bar{x}}\eta)_x - \eta_x\Big]\Big[(\sigma^{\bar{x}}\eta)_y - \eta_y\Big]\\
&= \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x.
\end{aligned}
$$

The proof of the third expression is analogous. □

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